MA140-Engineering Calculus

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National University of Ireland, Galway
Ollscoil na hÉireann, Gaillimh

September 1, 2018

- **Lectures:** 10 am, ENG-G018, Tuesday, Wednesday, Thursday
- **Lecturer:** Dr. Adib Makrooni
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- **Tutorials:** details will be announced later, normally they start two weeks after the first lecture
- **Supporting centre:** SUMS, visit the homepage to see the timetables: http://www.maths.nuigalway.ie/sums/
- **Text books:** Modern Engineering Mathematics by Glyn James, Thomas' Calculus or any basic calculus text book

Lecture 1

In this lecture we review real and complex numbers.

- Natural Numbers$=\mathbb{N} = \{1, 2, 3, \cdots\}$
  $1 + 3 = 4$, 4 is a natural number or 4 belongs to the set of natural numbers $\simeq 4 \in \mathbb{N}$

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  but $1-3=-2\notin\mathbb{N}_0$
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  but $1 - 3 = -2 \notin \mathbb{N}_0$
  $\mathbb{N}$ is a subset of $\mathbb{N}_0 \quad \longleftrightarrow \quad \mathbb{N} \subset \mathbb{N}_0$
  Remark:
  - $A \subset B$ it means for all elements in A or for any element in A like x then x is in $B \quad \longleftrightarrow \quad (\forall x \in A \quad \rightarrow x \in B)$
    So $\forall$ means "for all"
  - $A \not\subset B$ it means there exists an element in A like x which is not in $B \quad \longleftrightarrow \quad (\exists x \in A \quad .s.t \quad x \notin B)$

- Integers$=\mathbb{Z} = \{\cdots, -2, -1, 0, 1, 2, \cdots\}$
  $1 + (-4) = -3 \in \mathbb{Z}$ and $1 - 5 = -4 \in \mathbb{Z}$ so the addition and subtraction of two integers is again an integer but what about the division?
  $\frac{4}{2} = 2 \in \mathbb{Z}$, $(\frac{4}{3}) \notin \mathbb{Z}$

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- Rational Numbers$=\mathbb{Q}=\{\frac{a}{b} \quad | \quad a,b\in\mathbb{Z} \quad and \quad b\neq 0\}$
  for example $\frac{-3}{4}=-0.75000...$ or $\frac{1}{3}=-0.33333...$
  But $\sqrt{2}\notin\mathbb{Q}$

- Real Numbers=$\mathbb{R}$, the set of real numbers includes all the rational numbers and numbers like $\sqrt{2}, \pi = 3.14..., 0.11236$, these numbers are called irrational numbers so
  real numbers=(rational numbers)$\cup$(irrational numbers)
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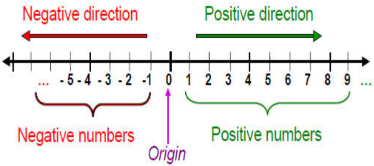

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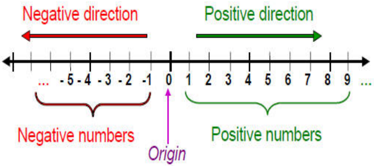

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- Complex Numbers=$\mathbb{C}$
If $c \in \mathbb{C}$, we can write $c = a + ib, \quad a, b \in \mathbb{R} \quad and \quad i = \sqrt{-1}$

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We represent a function in one of the two ways:

$$f : x \to y \quad or \quad y = f(x)$$

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Definition 1.1

A *function* from a set X to a set Y is a rule that assigns a unique (single) element $f(x) \in Y$ to each element $x \in X$.

Note: f associates each value of x in X with exactly one value of y in Y. It means we can not have different outputs for the same input.

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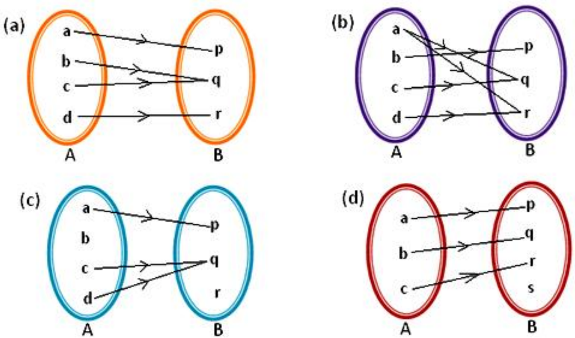

- when $y = f(x)$, y is said to be the *image* of x under f
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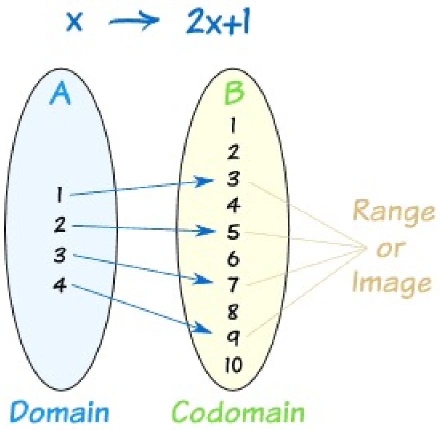
Note: It is not necessary for all elements of the codomain set Y to be images under f



Example 1.2

Identify the domain, codomain and range of

(a) $f(x) = 3x^2 + 1$

(b) $f(x) = \sqrt{(x+4)(3-x)}$

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solution (a):

$f(x)$ can be evaluated for all $x \in \mathbb{R}$, so domain=$\mathbb{R}$.
The lowest value $f(x)$ can take is 1 (when $x = 0$) so range=$[1, \infty]$.
We could write this as $\{y \mid y \geq 1 \ , \ y \in \mathbb{R}\}$.
We could take the codomain as $\mathbb{R}$ as it contains the range.

solution (b):

The domain is $[-4, 3]$, outside this range the function is not real valued i.e. it involves $\sqrt{-1}$.

The function takes value 0 at $x = -4$ and $x = 3$ and takes $7/2$, its highest value at $x = -1/2$. Therefore its range is $[0, 7/2]$. we can take the codomain to be $\mathbb{R}$ (as it contains the range)

A polynomial is a function of the form:

$$y = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

where $a_n, a_{n-1}, \cdots, a_0$ are constants.
These constants are called the *coefficients* of the polynomial.
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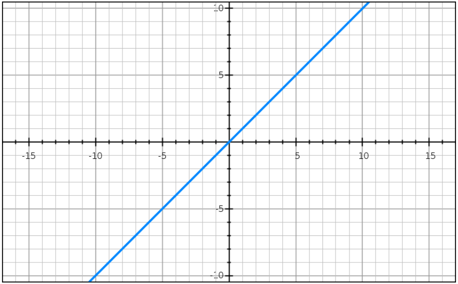

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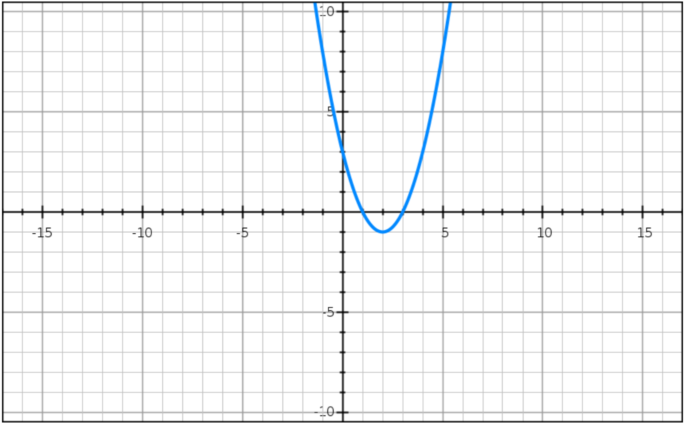

Example 1.5

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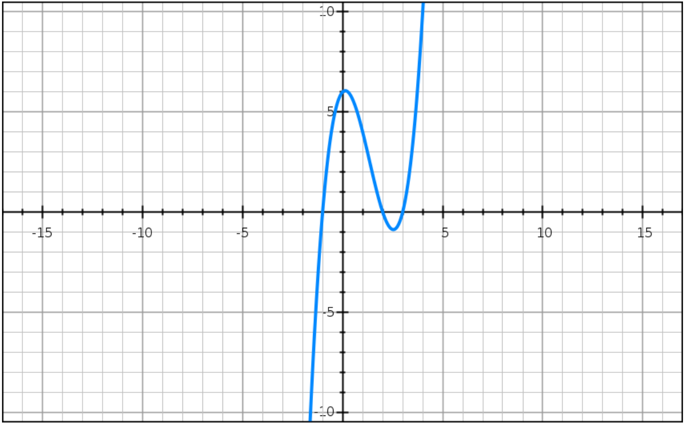

General facts on polynomial sketching:

A polynomial of degree n has up to $n-1$ bends.

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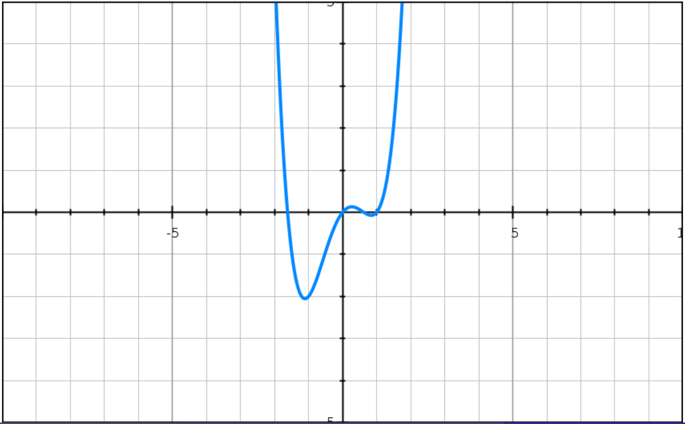
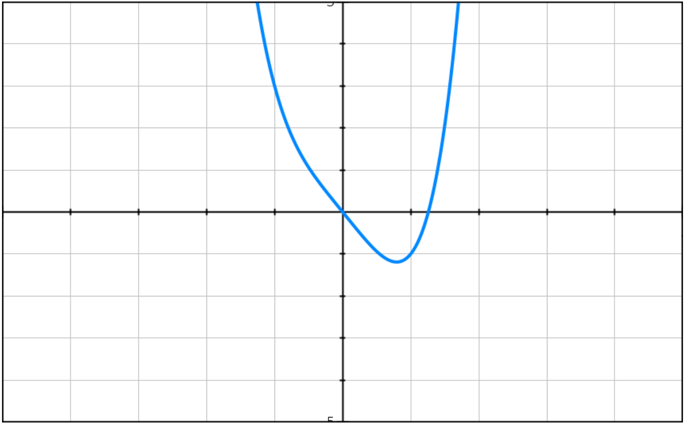
Example 1.6

a typical fourth degree polynomial has 3 bends. for example $y = x^4 - 2x^2 + x$



Example 1.7

the fourth degree polynomial $y = x^4 - 2x$ has only one bend.



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The $y-$intercept can be found by letting $x = 0$.
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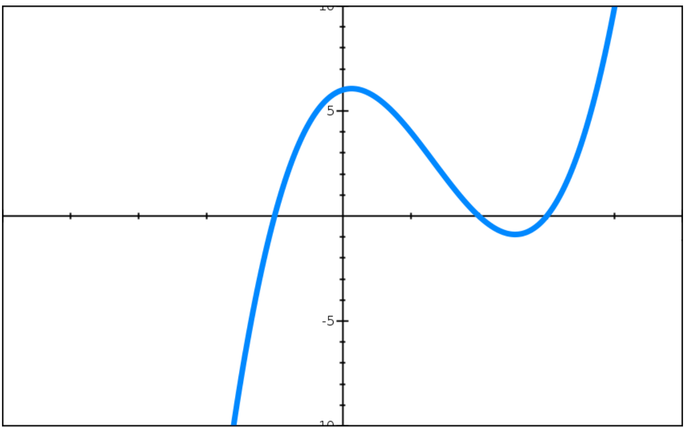
(The constant coefficient$=-$ product of the roots if the coefficient of the highest power$=1$)

By trial roots are $x = -1, 2, 3$


10
5
-5
-10

Definition 1.9

Rational functions have the general form

$$f(x) = \frac{p(x)}{q(x)}$$

where $p(x)$ and $q(x)$ are polynomials.

- **IF** degree of $p(x) <$ degree of $q(x)$, then $f(x)$ is a strictly proper rational function.
- **IF** degree of $p(x) =$ degree of $q(x)$, then $f(x)$ is a proper rational function.
- **IF** degree of $p(x) >$ degree of $q(x)$, then $f(x)$ is an improper rational function.

An improper or proper rational function can be expressed in terms of a strictly proper rational function

Example 1.10

Express $f(x) = \dfrac{3x^4 + 2x^3 - 5x^2 + 6x - 7}{x^2 - 2x + 3}$ in terms of a strictly proper rational function

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Example 1.10

Express $f(x) = \dfrac{3x^4 + 2x^3 - 5x^2 + 6x - 7}{x^2 - 2x + 3}$ in terms of a strictly proper rational function

$$f(x) = 3x^2 + 8x + 2 - \frac{14x + 13}{x^2 - 2x + 3}$$

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Example 1.11

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$$= \frac{x[A+B]+[2A-B]}{(x-1)(x+2)}$$

Compare $x[3] + [0]$ with $x[A + B] + [2A - B]$, so

$$\begin{cases} A + B &= 3 \\ 2A - B &= 0 \end{cases}$$

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From first equation $3 = 1 + B \quad \Rightarrow B = 2$
so

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Exercise:

Find the constants A, B and C, so that:

$$\frac{2x+1}{(x-2)(x+1)(x-3)}=\frac{A}{x-2}+\frac{B}{x+1}+\frac{C}{x-3}$$

2. Linear Factors to Powers Greater than 1 (i.e. repeated linear factors):

If $(x-\alpha)^k$ appears in the denominator, it will give rise to the following terms:

$$\frac{A_1}{x-\alpha}+\frac{A_2}{(x-\alpha)^2}+\cdots+\frac{A_k}{(x-\alpha)^k}$$

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Exercise:

Show from the start

$$\frac{3x+1}{(x-1)^2(x+2)}=\frac{5/9}{x-1}+\frac{4/3}{(x-1)^2}+\frac{-5/9}{x+2}$$

3. Irreducible quadratic factors:

Irreducible quadratic factors can not be factorised using real numbers e.g. $x^2 + x + 1$

An irreducible quadratic $ax^2 + bx + c$ gives rise to partial fractions of the form

$$\frac{Ax + B}{ax^2 + bx + c}$$

Example 1.12

Express the following in partial fractions:

$$\frac{5x}{(x^2 + x + 1)(x - 2)}$$

$$\frac{5x}{(x^2+x+1)(x-2)} = \frac{Ax+B}{x^2+x+1} + \frac{C}{x-2}$$

$$\frac{5x}{(x^2+x+1)(x-2)} = \frac{Ax+B}{x^2+x+1} + \frac{C}{x-2}$$

$$= \frac{(Ax+B)(x-2)+C(x^2+x+1)}{(x^2+x+1)(x-2)}$$

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Therefore: $5x = (Ax+B)(x-2)+C(x^2+x+1)$

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Therefore: $5x = (Ax+B)(x-2)+C(x^2+x+1)$
Now let $x=2$ then

$$5(2) = 0 + C(2^2+2+1)$$

$$\begin{aligned} \Rightarrow &\quad 10 = C(7) \\ \Rightarrow &\quad C = 10/7 \end{aligned}$$

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constant term on the RHS=0 so

$$0 = -2B + C$$

$$\Rightarrow 2B = C$$
$$\Rightarrow B = C/2 = 5/7$$

The coefficient of $x^2 = 0$ so $0 = A + C \quad \Rightarrow \quad A = -C = -10/7$, so

$$\frac{5x}{(x^2+x+1)(x-2)} = \frac{\frac{-10}{7}x + \frac{5}{7}}{x^2+x+1} + \frac{\frac{10}{7}}{x-2}$$

4. Irreducible quadratic factors to powers greater than 1:

If $ax^2 + bx + c$ is an irreducible quadratic factor, then $(ax^2 + bx + c)^k$ in the denominator gives rise to

$$\frac{A_1x + B_1}{ax^2 + bx + c} + \frac{A_2x + B_2}{(ax^2 + bx + c)^2} + \cdots + \frac{A_kx + B_k}{(ax^2 + bx + c)^k}$$

Example 1.13

What is the difference between these two functions:

$$\frac{x^2 - 1}{x - 1} \quad and \quad x + 1$$

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$$\frac{x^2-1}{x-1} = \frac{(x-1)(x+1)}{x-1}$$

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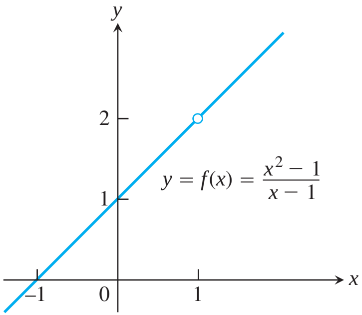
$$\frac{x^2-1}{x-1} = x+1 \quad for \quad x \neq 1$$

We have seen that the function f equals $x+1$ except at $x=1$, so the graph of f is thus the line $x+1$ with the point $(1,2)$ removed.



Example 1.14

How does the function

$$f(x) = \frac{x^2 - 1}{x - 1}$$

behave near $x = 1$

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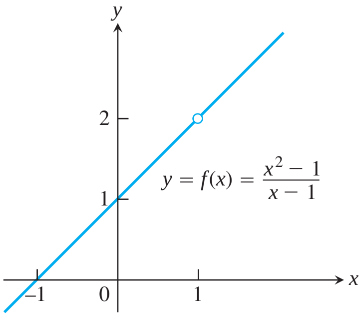

Values of x below and above 1	$f(x) = \dfrac{x^2 - 1}{x - 1} = x + 1, \quad x \neq 1$
0.9	1.9
1.1	2.1
0.99	1.99
1.01	2.01
0.999	1.999
1.001	2.001
0.999999	1.999999
1.000001	2.000001

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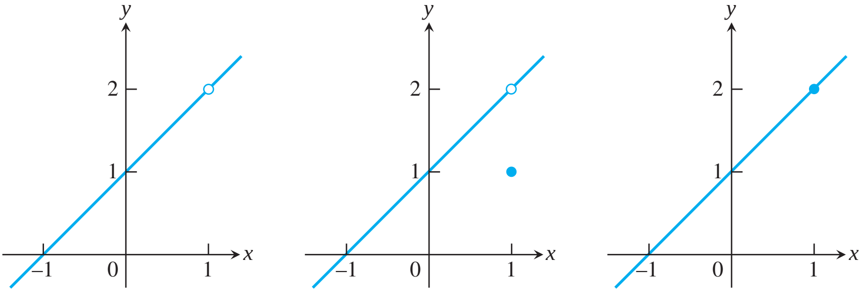
1.000001	2.000001

$$\lim_{x \to 1} f(x) = \lim_{x \to 1} \frac{x^2 - 1}{x - 1} = 2$$



(a) $f(x) = \dfrac{x^2 - 1}{x - 1}$

(b) $g(x) = \begin{cases} \dfrac{x^2 - 1}{x - 1}, & x \neq 1 \\ 1, & x = 1 \end{cases}$

(c) $h(x) = x + 1$

Finding the Limits

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Sometimes $\lim_{x\to a} f(x)$ can be evaluated by calculating $f(a)$. This holds, for example, whenever $f(x)$ is an algebraic combination of polynomials and trigonometric functions for which $f(a)$ is defined.

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Note: The Identity and Constant Functions Have Limits at Every Point.

The Limit Laws

If L, M, a and k are real numbers and

$$\lim_{x \to a} f(x) = L \quad and \quad \lim_{x \to a} g(x) = M$$

then:

- *Sum Rule*: The limit of the sum of two functions is the sum of their limits.

$$\lim_{x \to a}[f(x) + g(x)] = \lim_{x \to a} f(x) + \lim_{x \to a} g(x) = L + M$$

- *Difference Rule:* The limit of the difference of two functions is the difference of their limits.

$$\lim_{x \to a}[f(x) - g(x)] = \lim_{x \to a} f(x) - \lim_{x \to a} g(x) = L - M$$

- *Product Rule:* The limit of a product of two functions is the product of their limits.

$$\lim_{x\to a}[f(x)\cdot g(x)] = (\lim_{x\to a} f(x))\cdot(\lim_{x\to a} g(x)) = L\cdot M$$

- *Constant Multiple Rule:* The limit of a constant times a function is the constant times the limit of the function.

$$\lim_{x\to a}(k\cdot f(x)) = k\cdot(\lim_{x\to a} f(x) = k\cdot L$$

- *Quotient Rule:* The limit of a quotient of two functions is the quotient of their limits, provided the limit of the denominator is not zero.

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} = \frac{L}{M}, \quad M\neq 0$$

- *Power Rule:* If r and s are integers with no common factor and $s \neq 0$, then
$$\lim_{x\to a}(f(x))^{r/s} = (\lim_{x\to a} f(x))^{r/s} = L^{r/s}$$

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Example 1.15

- $\lim_{x\to 1}(x^3 + 4x^2 - 3) = \lim_{x\to 1} x^3 + \lim_{x\to 1} 4x^2 - \lim_{x\to 1} 3 = 1 + 4 - 3 = 2$

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Example 1.16

- $\displaystyle \lim_{x\to 1} \frac{x^4 + x^2 - 1}{x^2 + 5} = \frac{\lim_{x\to 1}(x^4 + x^2 - 1)}{\lim_{x\to 1}(x^2 + 5)} = \frac{1}{6}$

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Example 1.17

- $\lim_{x\to 1} \sqrt{4x^2 - 3} = \sqrt{\lim_{x\to 1}(4x^2 - 3)} = 1$

Example 1.18

Evaluate:

$$\lim_{x \to 1} \frac{x^2 + x - 2}{x^2 - x}$$

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$$\lim_{x\to 1}\frac{x^2+x-2}{x^2-x}$$

$$\lim_{x\to 1}\frac{x^2+x-2}{x^2-x}=\lim_{x\to 1}\frac{(x-1)(x+2)}{x(x-1)}=\lim_{x\to 1}\frac{x+2}{x}=3$$

Example 1.18

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$$\lim_{x \to 1} \frac{x^2 + x - 2}{x^2 - x} = \lim_{x \to 1} \frac{(x-1)(x+2)}{x(x-1)} = \lim_{x \to 1} \frac{x+2}{x} = 3$$

Note: If the denominator is zero, canceling common factors in the numerator and denominator may reduce the fraction to one whose denominator is no longer zero

Example 1.19

$$\lim_{x\to 2}\left(\frac{1}{2}-\frac{1}{x}\right)\frac{1}{x-2}$$

$$\begin{aligned}\lim_{x\to 2}\left(\frac{1}{2}-\frac{1}{x}\right)\frac{1}{x-2} &= \lim_{x\to 2}\frac{x-2}{2x}\cdot\frac{1}{x-2}\\ &= \lim_{x\to 2}\frac{1}{2x}=\frac{1}{2(2)}=\frac{1}{4}\end{aligned}$$

Example 1.20

Evaluate

$$\lim_{x\to 0} \frac{\sqrt{1+x^2}-1}{x^2}$$

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$$\lim_{x\to 0} \frac{(1+x^2-1)}{x^2(\sqrt{1+x^2}+1)} = \lim_{x\to 0} \frac{(x^2)}{x^2(\sqrt{1+x^2}+1)}$$

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Evaluate

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$$\lim_{x \to 0} \frac{(1+x^2-1)}{x^2(\sqrt{1+x^2}+1)} = \lim_{x \to 0} \frac{(x^2)}{x^2(\sqrt{1+x^2}+1)}$$

$$\lim_{x \to 0} \frac{1}{\sqrt{1+x^2}+1} = \frac{1}{\sqrt{1}+1} = 1/2$$

Theorem 1.21

Sandwich Theorem Suppose that $g(x) \leq f(x) \leq h(x)$ *for all* x *in some open interval containing* c, *except possibly at* c *itself. Suppose also that*

$$\lim_{x \to c} g(x) = \lim_{x \to c} h(x) = L$$

Then $\lim_{x \to c} f(x) = L$

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Sandwich Theorem Suppose that $g(x) \le f(x) \le h(x)$ *for all* x *in some open interval containing* c*, except possibly at* c *itself. Suppose also that*

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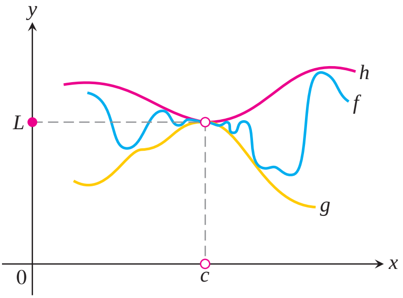

Example 1.22

Given

$$1-\frac{x^2}{4} \le u(x) \le 1+\frac{x^2}{2} \quad forall \quad x \neq 0$$

Find the $\lim_{x\to 0} u(x)$, no matter how complicated u is.

Example 1.22

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$$1 - \frac{x^2}{4} \leq u(x) \leq 1 + \frac{x^2}{2} \quad forall \quad x \neq 0$$

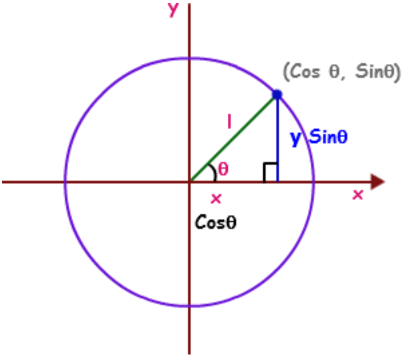
Find the $\lim_{x\to 0} u(x)$, no matter how complicated u is.

Since

$$\lim_{x\to 0}(1 - \frac{x^2}{4}) = 1 \quad and \quad \lim_{x\to 0}(1 + \frac{x^2}{2}) = 1$$

the Sandwich theorem implies that

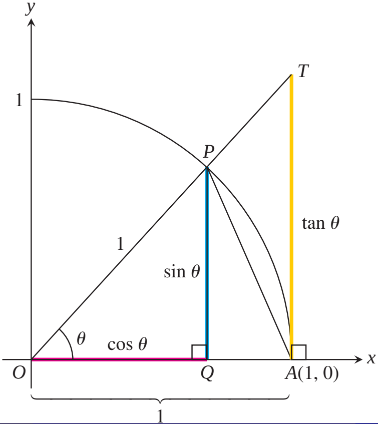
$$\lim_{x\to 0} u(x) = 1$$


y
(Cos θ, Sinθ)
1
y Sinθ
θ
x
Cosθ
x

Example 1.23

An important limit:

$$\lim_{x\to 0} \frac{\sin x}{x} = 1$$



Area of $\triangle OAP <$ Area of the sector $OAP <$ Area of $\triangle OAT$
Area of $\triangle OAP = \frac{1}{2}\sin\theta$
Area of the sector $OAP = \frac{1}{2}r^2\theta = \frac{\theta}{2}$
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Taking the reciprocals reverses the inequalities so:

$$1 > \frac{\sin\theta}{\theta} > \cos\theta$$

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Since $\lim_{\theta\to 0}\cos\theta = \lim_{\theta\to 0} 1 = 1$, The Sandwich theorem gives:

$$\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1$$

Example 1.24

$$\lim_{x \to 0} \frac{\tan 3x}{\sin 2x}$$

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$$\begin{aligned}
\lim_{x \to 0} \frac{\tan 3x}{\sin 2x} &= \lim_{x \to 0} \frac{\sin 3x}{\cos 3x} \cdot \frac{1}{\sin 2x} \\
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&= \lim_{x\to 0} \frac{\sin 3x}{3x} \cdot \frac{1}{\cos 3x} \cdot \frac{2x}{\sin 2x} \cdot \frac{3x}{2x}
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$$= (1)(1)(1)(3/2) = 3/2$$

Example 1.25

$$\lim_{x\to 0} \frac{1-\cos x}{x^2}$$

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Note: $\cos(2A) = 1 - 2\sin^2(A)$

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$$= (1)(1)(1/2) = 1/2$$

Useful approximation:

1. when $x \to 0$ then $\cos x \approx 1 - \frac{x^2}{2}$
2. when $x \to 0$ then $\sin x \approx x - \frac{x^3}{6}$

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Example 1.26

$$\lim_{x\to 0} \frac{\sin x - x}{x^3}$$

$$\lim_{x\to 0} \frac{\sin x - x}{x^3} = \lim_{x\to 0} \frac{[x - \frac{x^3}{6}] - x}{x^3}$$

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$$\lim_{x \to 0} \frac{\sin x - x}{x^3} = \lim_{x \to 0} \frac{[x - \frac{x^3}{6}] - x}{x^3}$$

$$= \lim_{x \to 0} \frac{-\frac{x^3}{6}}{x^3} = -1/6$$

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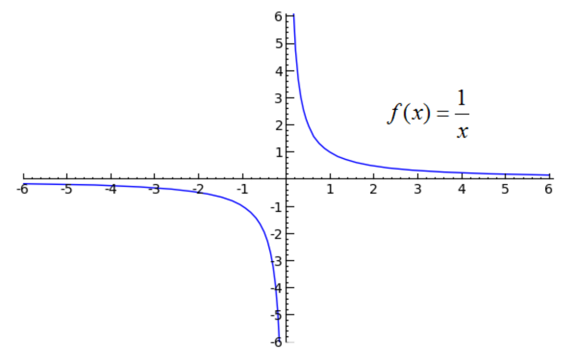

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$$\frac{infinity}{constant \neq 0} \quad or \quad \frac{constant \neq 0}{0} \Rightarrow \infty$$

Example 1.27

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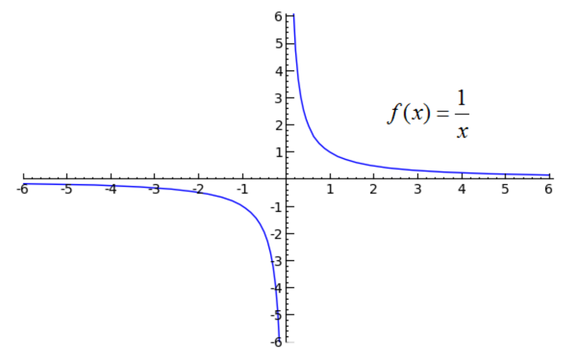


Example 1.28

If $f(x) = \frac{x+1}{x^3}$, find $\lim_{x \to 0} f(x)$

$f(x)$ tends to $\frac{constant \approx 1}{\approx 0}$
so $\lim_{x \to 0} f(x) = \infty$, it does not exist.

Example 1.28

If $f(x) = \frac{x+1}{x^3}$, find $\lim_{x\to 0} f(x)$

$f(x)$ tends to $\frac{constant \approx 1}{\approx 0}$
so $\lim_{x\to 0} f(x) = \infty$, it does not exist.

Note:

$$\frac{constant \neq 0}{infinity} \quad or \quad \frac{0}{constant \neq 0} \quad \Rightarrow 0$$

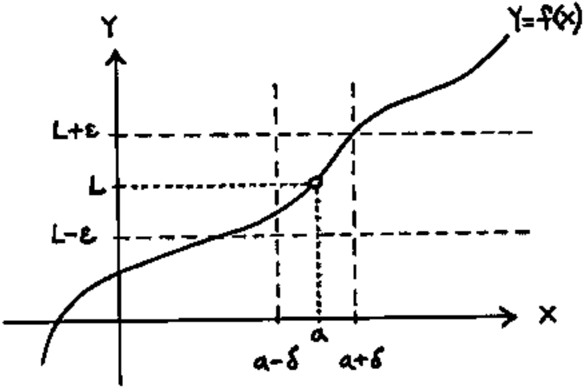
one can think of sequences as values of continuous functions at
$x = 1, 2, 3, \cdots$
for example $f(x) = x^2 + 1$, $a_n = n^2 + 1$
$\{a_n\} = \{2, 5, 10, 17, \cdots\}$
we can use our limit theory on continuous functions to decide if a sequence has a limit.

Example 1.29

Does $\{a_n\}$ with $a_n = \frac{n^2+1}{n+1}$ have a limit.

$\lim_{n\to\infty} \frac{n^2+1}{n+1} = \lim_{n\to\infty} \frac{n+\frac{1}{n}}{1+\frac{1}{n}} = \infty$

$\{a_n\}$ has no limit if diverges to infinity



- a is a fixed number which x can take i.a. $x = a$
- δ and ϵ are small positive numbers i.e. $\delta > 0$
- $|x - a| < \delta$ the distance from x to a is less than δ
- $f(x)$ approaches a number l as x approaches a

So if I am able to find a δ given an ϵ then (in an informal way) this means: that we can make the value of $f(x)$ as close as we like to L by taking x sufficiently close to a.

Formally:

A function $f(x)$ is said to approach a limit l as x approaches the value a, if given any small positive number ϵ, it is possible to find a positive number δ, such that for any x:

$$|x-a|<\delta \quad \Rightarrow \quad |f(x)-l|<\epsilon$$

we write

$$\lim_{x\to a} f(x) = l$$

Example 1.30

Prove formally that:

$$\lim_{x \to 3} (4x - 5) = 7$$

Given ϵ, then

$$|f(x) - l| < \epsilon \Leftrightarrow |(4x - 5) - 7| < \epsilon$$

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$$\Leftrightarrow |4x - 12| < \epsilon$$

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$$\Leftrightarrow |4x-12|<\epsilon$$

$$\Leftrightarrow 4|x-3|<\epsilon$$

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$$\Leftrightarrow |4x - 5 - 7| < \epsilon$$

$$\Leftrightarrow |4x - 12| < \epsilon$$

$$\Leftrightarrow 4|x - 3| < \epsilon$$

$$\Leftrightarrow |x - 3| < \epsilon/4$$

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Prove formally that:

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Given ϵ, then

$$|f(x) - l| < \epsilon \Leftrightarrow |(4x - 5) - 7| < \epsilon$$

$$\Leftrightarrow |4x - 5 - 7| < \epsilon$$

$$\Leftrightarrow |4x - 12| < \epsilon$$

$$\Leftrightarrow 4|x - 3| < \epsilon$$

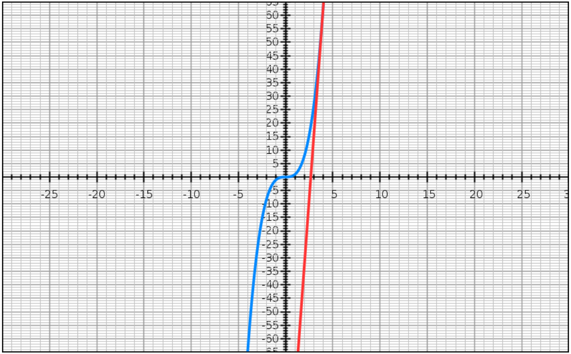
$$\Leftrightarrow |x - 3| < \epsilon/4$$

so if we pick $\delta = \epsilon/4$ then from above we see that if $|x - 3| < \delta = \epsilon/4$ implies δ so $|(4x - 5) - 7| < \epsilon$.

limits have practical uses.

Example 1.31

Using limits find the slope of the tangent to the curve $y = x^3$ at $x = 4$



$(4, 4^3)$ is a point on $y = x^3$, $(4th, (4th)^3)$ is a point on $y = x^3$.
The slope of a line through the points (x_1, y_1) and (x_2, y_2) is

$$\frac{y_2 - y_1}{x_2 - x_1}$$

so the slope through the two points above is
$\frac{(4+h)^3-4^3}{(4+h)-4} = \frac{4^3+3h4^2+3h^24+h^3-4^3}{h}$
$\frac{48h+12h^2+h^3}{h} = 48 + 12h + h^2$
to find the slope pf the tangent at $(4, 4^3)$, we find the limit as $h \to 0$ of the above i.e. $\lim_{h\to 0}(48 + 12h + h^2) = 48$

Another practical use of limits

Example 1.32

find the area between the x-axis and the curve $y = x^3$ between $x = 0$ and $x = 1$

Exercises

(1)

$$\lim_{x \to 2} \frac{x^3 - 8}{x^2 - 4}$$

(2)

$$\lim_{x \to 1} \frac{x^2 - 1}{(\sqrt{x} - 1)(x + 2)}$$

(3)

$$\lim_{x \to 0} \frac{\sin^2 x}{\sin 2x \cdot \tan 4x}$$

(4)

$$\lim_{x \to 0} x \cos \frac{1}{x^2}$$

Example 1.33

Prove formally that

$$\lim_{x \to 7} \frac{-x^2 + 9x - 14}{x - 7} = -5$$

Example 1.33

Prove formally that

$$\lim_{x\to 7} \frac{-x^2+9x-14}{x-7} = -5$$

Given ϵ, then

$$|f(x)-l| < \epsilon \Leftrightarrow |\frac{-x^2+9x-14}{x-7} - (-5)| < \epsilon$$

$$\Leftrightarrow |\frac{-(x-7)(x-2)}{x-7} + 5| < \epsilon$$

$$\Leftrightarrow |-(x-2)+5| < \epsilon$$

$$\Leftrightarrow |-x+7| < \epsilon \Leftrightarrow |x-7| < \epsilon$$

so we should pick $\epsilon = \delta$

- $|x-a|<\delta$ is the same as $a-\delta<x<a+\delta$ or $-\delta<x-a<\delta$

Example 1.34

solve

$$|x-7|<3$$

$$-3<x-7<3 \Leftrightarrow 7-3<x<7+3 \Leftrightarrow 4<x<10$$

- For $x \in \mathbb{R}$:

$$|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

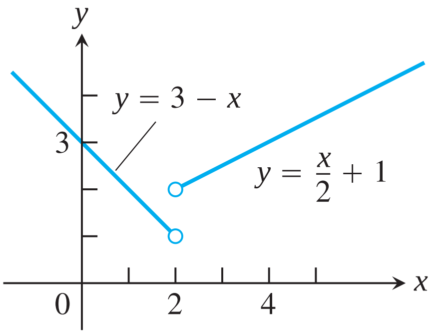
- When multiplying an inequality by a negative number flip the inequality sign.

One-sided limits

Example 1.35

Let

$$f(x) = \begin{cases} 3 - x, & x < 2 \\ \frac{x}{2} + 1, & x > 2 \end{cases}$$



Note:

- The function approaches 1 as x approaches 2 from the left.
- The function approaches 2 as x approaches 2 from the right.

Notation:

$$\lim_{x \to 2^-} f(x) = 1 \quad \lim_{x \to 2^+} f(x) = 2$$

These one-sided limits can be defined formally using the ϵ/δ notation.
Clearly: $\lim_{x\to 2} f(x)$ above does not exist.

Theorem 1.36

A function $f(x)$ has a limit as x approaches a if and only if it has left-handed and right-handed limits there and these one-sided limits are equal.

$$\lim_{x\to a} f(x) = l \quad \Leftrightarrow \quad \lim_{x\to a^-} f(x) = l \quad \textit{and} \quad \lim_{x\to a^+} f(x) = l$$

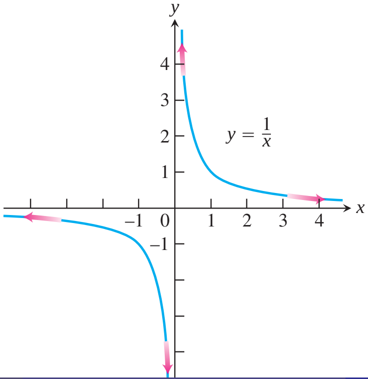
Example 1.37

$$\lim_{x\to 0^+} \sqrt{x} = 0$$

Limits at Infinity

Example 1.38

Find the limit of $y = \frac{1}{x}$ as $x \to \pm\infty$



Asymptotes

Example 1.39

$$\lim_{x\to\infty} \frac{x+3}{x+2}$$

Asymptotes

Example 1.39

$$\lim_{x\to\infty} \frac{x+3}{x+2}$$

When dealing with limits at infinity of rational functions it can be useful to divide top and bottom by the highest power of the bottom.

$$\lim_{x\to\infty} \frac{x+3}{x+2} = \lim_{x\to\infty} \frac{x(1+\frac{3}{x})}{x(1+\frac{2}{x})} = \lim_{x\to\infty} \frac{(1+\frac{3}{x})}{(1+\frac{2}{x})} = 1$$

this exercise tells us that when x get very big, the function tends to 1.

We say that the function $\dfrac{x+3}{x+2}$ has a *horizontal asymptote* $y = 1$

Definition 1.40

A line $y = b$ is a horizontal asymptote of the graph of a function $y = f(x)$ if either

$$\lim_{x\to\infty} f(x) = b \quad or \quad \lim_{x\to-\infty} f(x) = b$$

Example 1.41

Find the horizontal asymptote of $f(x) = \frac{5x^2+8x-3}{3x^2+2}$

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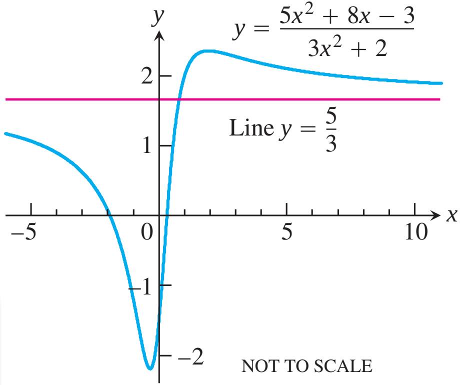
Find the horizontal asymptote of $f(x) = \frac{5x^2+8x-3}{3x^2+2}$

$$\lim_{x\to\infty} \frac{5x^2+8x-3}{3x^2+2} = \lim_{x\to\infty} \frac{5+(8/x)-(3/x^2)}{3+(2/x^2)} = \frac{5+0-0}{3+0} = \frac{5}{3}$$

Example 1.41

Find the horizontal asymptote of $f(x) = \frac{5x^2+8x-3}{3x^2+2}$

$$\lim_{x\to\infty} \frac{5x^2+8x-3}{3x^2+2} = \lim_{x\to\infty} \frac{5+(8/x)-(3/x^2)}{3+(2/x^2)} = \frac{5+0-0}{3+0} = \frac{5}{3}$$



Note that the denominator of $\dfrac{x+3}{x+2}$ is zero when $x = -2$.

$$\lim_{x \to -2^-} \frac{x+3}{x+2} = -\infty \qquad \lim_{x \to -2^+} \frac{x+3}{x+2} = \infty$$

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We say that the function $\dfrac{x+3}{x+2}$ has a *vertical asymptote* $x = -2$

Note that the denominator of $\dfrac{x+3}{x+2}$ is zero when $x=-2$.

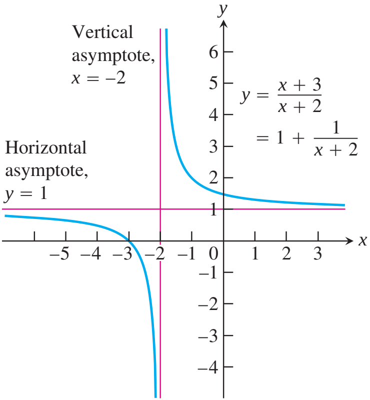
$$\lim_{x\to -2^-}\frac{x+3}{x+2}=-\infty \qquad \lim_{x\to -2^+}\frac{x+3}{x+2}=\infty$$

We say that the function $\dfrac{x+3}{x+2}$ has a *vertical asymptote* $x=-2$

Definition 1.42

A line $x=a$ is a vertical asymptote of the graph of a function $y=f(x)$ if either

$$\lim_{x\to a^+} f(x)=\infty \quad or \quad \lim_{x\to a^-} f(x)=\infty$$



y
x
Vertical asymptote, $x = -2$
Horizontal asymptote, $y = 1$
$y = \frac{x+3}{x+2}$
$= 1 + \frac{1}{x+2}$
6
5
4
3
2
1
0
-1
-2
-3
-4
-5
1
2
3

Example 1.43

Find all asymptotes of $f(x)$ and plot

$$f(x) = \frac{x^2 - 3}{2x - 4}$$

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$$f(x) = \frac{x^2 - 3}{2x - 4}$$

writing this as a strictly proper rational function

$$f(x) = \frac{x}{2} + 1 + \frac{1}{2x - 4}$$

Example 1.43

Find all asymptotes of $f(x)$ and plot

$$f(x) = \frac{x^2 - 3}{2x - 4}$$

writing this as a strictly proper rational function

$$f(x) = \frac{x}{2} + 1 + \frac{1}{2x - 4}$$

so, when x gets very big $f(x)$ tends to $\frac{x}{2} + 1$.
Also when x gets very negative, $f(x)$ tends to $\frac{x}{2} + 1$.

Example 1.43

Find all asymptotes of $f(x)$ and plot

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writing this as a strictly proper rational function

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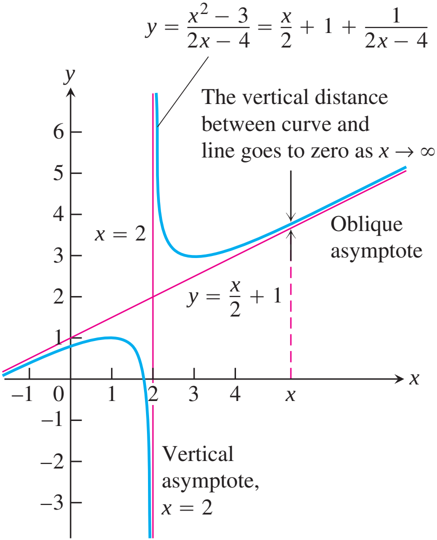
so, when x gets very big $f(x)$ tends to $\frac{x}{2} + 1$.
Also when x gets very negative, $f(x)$ tends to $\frac{x}{2} + 1$.
when $2x - 4 = 0$ or $x = 2$, we have a vertical asymptote.
the zeros of the function are:

$$x^2 - 3 = 0 \Rightarrow x = \pm 3$$



$y = \frac{x^2 - 3}{2x - 4} = \frac{x}{2} + 1 + \frac{1}{2x - 4}$
The vertical distance between curve and line goes to zero as $x \to \infty$
Oblique asymptote
$x = 2$
$y = \frac{x}{2} + 1$
Vertical asymptote, $x = 2$
y
x
6
5
4
3
2
1
0
–1
–2
–3
1
2
3
4

Example 1.44

Find the horizontal and vertical asymptotes of the graph of

$$f(x) = -\frac{8}{x^2 - 4}$$

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First, since $\lim_{x\to\infty} f(x) = 0$, the line $y = 0$ is a horizontal asymptote.

Example 1.44

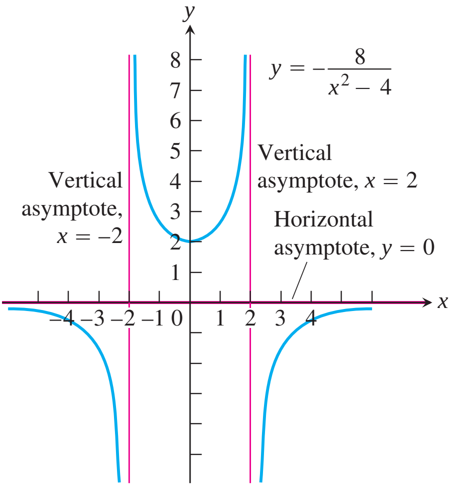
Find the horizontal and vertical asymptotes of the graph of

$$f(x) = -\frac{8}{x^2 - 4}$$

First, since $\lim_{x\to\infty} f(x) = 0$, the line $y = 0$ is a horizontal asymptote. Also, since

$$\lim_{x\to 2^+} f(x) = -\infty \quad and \quad \lim_{x\to 2^-} f(x) = \infty$$

the line $x = 2$ is a vertical asymptote both from the right and from the left.



$y = -\frac{8}{x^2 - 4}$
Vertical asymptote, $x = -2$
Vertical asymptote, $x = 2$
Horizontal asymptote, $y = 0$
y
x
8
7
6
5
4
3
2
1
−4 −3 −2 −1 0 1 2 3 4

Exercises

- $$\lim_{x\to 2^-} \frac{x-3}{x^2-4}$$
- $$\lim_{x\to 2^-} \frac{x-3}{x^2-4}$$
- $$\lim_{x\to 2} \frac{2-x}{(x-2)^3}$$

A continuous function is one with an "unbroken" graph

A continuous function is one with an "unbroken" graph

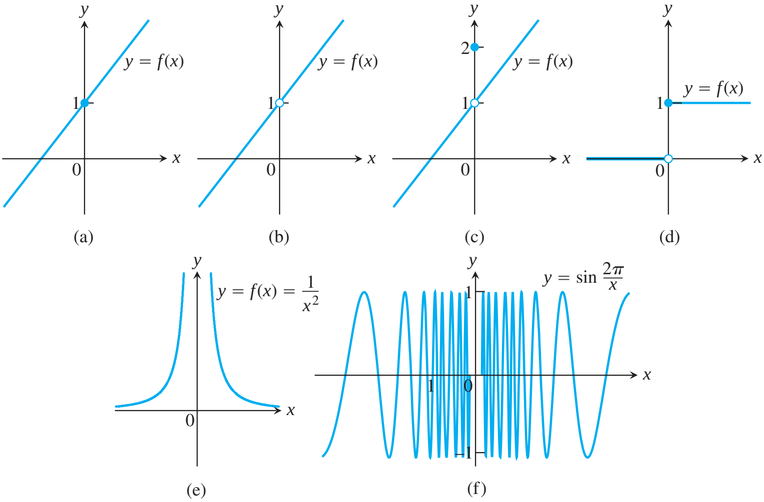
Definition 1.45

A function f is continuous at $x = a$ if:

- the point a is in the domain of f
- $f(x)$ has a limit as $x \to a$
- $\lim_{x\to a} f(x) = f(a)$

If f is continuous at every point in its domain, we say f is continuous.
Note: Lots of functions are continuous e.g. polynomials, trigonometric (not $\tan$), $|x|$ and so on.

Here are some examples of discontinuity.



Example 1.46

Consider the function

$$f(x) = \begin{cases} x+1, & x < 2 \\ bx^2, & x \geqslant 2 \end{cases}$$

For what values of b is f continuous at $x = 2$.

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For what values of b is f continuous at $x = 2$.

We have

$$\lim_{x \to 2^-} f(x) = 2 + 1 = 3$$

and

$$\lim_{x \to 2^+} f(x) = b(2)^2 = 4b$$

Example 1.46

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Also note

$$f(2) = b(2)^2 = 4b$$

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and

$$\lim_{x \to 2^+} f(x) = b(2)^2 = 4b$$

Also note

$$f(2) = b(2)^2 = 4b$$

First, for the function to have a limit at $x = 2$,

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^+} f(x)$$

so $3 = 4b$, then $b = 3/4$, so

$$\lim_{x \to 2} f(x) = 3$$

so $3 = 4b$, then $b = 3/4$, so

$$\lim_{x \to 2} f(x) = 3$$

Second, we will check that this value of b ensures

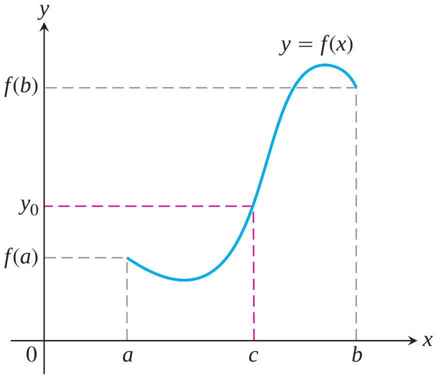
$$\lim_{x \to 2} f(x) = f(2)$$

$3 = 4b$ or $3 = 4(3/4)$. So when $b = 3/4$, $f(x)$ is continuous at $x = 2$.

(The Intermediate Value Theorem)

Theorem 1.47

A function $y = f(x)$ that is continuous on an interval $[a, b]$ takes on every value between $f(a)$ and $f(b)$. In other words, if y_0 is any value between $f(a)$ and $f(b)$, then $y_0 = f(c)$ for some c in $[a, b]$



Example 1.48

Sketch a discontinuous graph for which the above theorem does not hold

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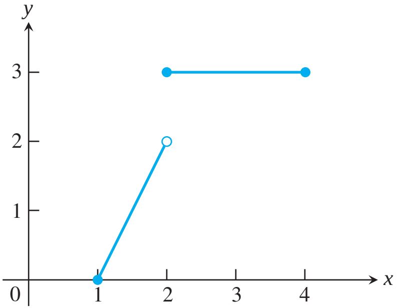
Sketch a discontinuous graph for which the above theorem does not hold

The function

$$f(x) = \begin{cases} 2x-2, & 1 \leqslant x < 2 \\ 3, & 2 \leqslant x \leqslant 4 \end{cases}$$

This function does not take on all values between $f(1) = 0$ and $f(4) = 3$, it misses all the values between 2 and 3.



Example 1.49

Show that there is a root of the equation

$$4x^3 - 6x^2 + 3x - 2 = 0$$

between 1 and 2.

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Let $f(x) = 4x^3 - 6x^2 + 3x - 2$, we will use the above theorem with $a = 1, b = 2, c = 0$, so

Example 1.49

Show that there is a root of the equation

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between 1 and 2.

Let $f(x) = 4x^3 - 6x^2 + 3x - 2$, we will use the above theorem with $a = 1, b = 2, c = 0$, so

- $f(x)$ is a polynomial so it is continuous
- $f(1) = 4 - 6 + 3 - 2 = -1 < 0$
  $f(2) = 32 - 24 + 6 - 2 = 12 > 0$
- and $f(a) < c < f(b)$

All conditions hold so, there exists a y, $1 < y < 2$ such that $f(y) = 0$
So there is a root between 1 and 2

Example 1.50

How many roots does $x^3 + 1 = 3x^2$ have?

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So,

$$x^3 - 3x^2 + 1 = 0$$

We define

$$f(x) = x^3 - 3x^2 + 1$$

$f(x)$ is a polynomial and hence continuous.

Example 1.50

How many roots does $x^3 + 1 = 3x^2$ have?

So,

$$x^3 - 3x^2 + 1 = 0$$

We define

$$f(x) = x^3 - 3x^2 + 1$$

$f(x)$ is a polynomial and hence continuous.

- $f(-1) = -1 - 3 + 1 < 0$ negative
- $f(0) = 1 > 0$ positive
- $f(2) = 8 - 3(4) + 1 < 0$ negative
- $f(3) = 1 > 0$ positive

So using The Intermediate Value Theorem, we see that $f(x) = 0$ has 3 roots.

Example 1.51

Find the number of asymptotes and discontinuities of

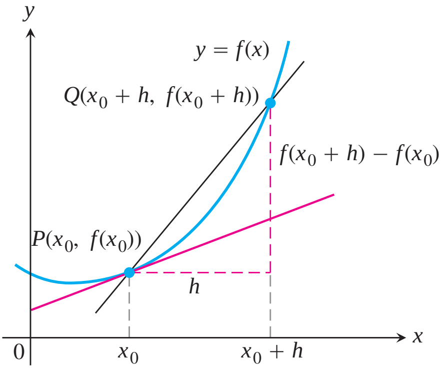
$$f(x) = \frac{(x-1)(x-2)}{(x+1)(x-3)}$$

Then find the $x-$intercept and $y-$intercept and plot f

Example 1.52

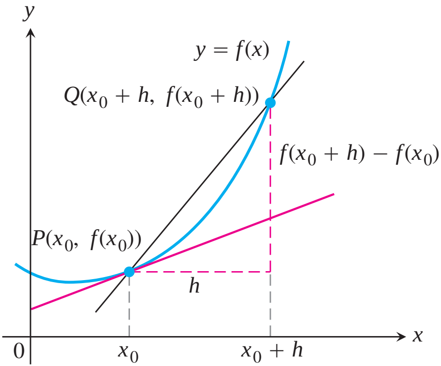
How many discontinuities has

$$f(x) = \begin{cases} \frac{(x-1)(x-2)}{(x+1)(x-3)}, & x \neq 1 \\ 1, & x = 1 \end{cases}$$

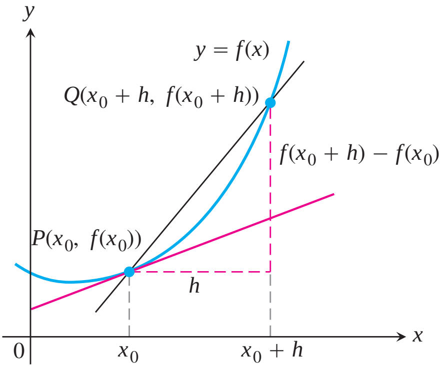


y
$y = f(x)$
$Q(x_0 + h,\ f(x_0 + h))$
$f(x_0 + h) - f(x_0)$
$P(x_0,\ f(x_0))$
h
0
x_0
$x_0 + h$
x

The change in distance $= f(x_0 + h) - f(x_0)$

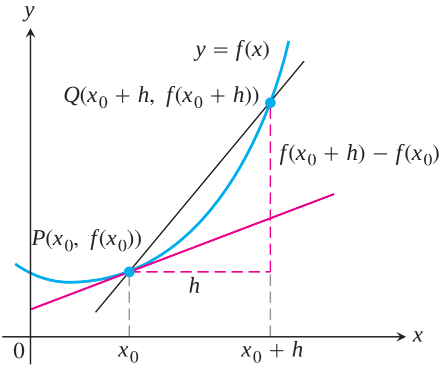


The change in distance $= f(x_0 + h) - f(x_0)$
The change in time $= (x_0 + h) - x_0 = h$



The change in distance $= f(x_0 + h) - f(x_0)$
The change in time $= (x_0 + h) - x_0 = h$
The average speed between P and Q is

$$\frac{f(x_0 + h) - f(x_0)}{h}$$



The slope of the curve $y = f(x)$ at the point $P(x_0, f(x_0))$ is the number

$$\lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

We call this limit the derivative of f at x_0.

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$$\lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

We call this limit the derivative of f at x_0.

Definition 1.53

The derivative of the function $f(x)$ with respect to the variable x is the function f' or $\frac{df}{dx}$ whose value at x is

$$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$$

Example 1.54

Use the above definition to find the derivative of $f(x) = x^2$

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We know that

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Here $f(x+h) = (x+h)^2 = x^2 + h^2 + 2hx$ so

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{(x^2 + h^2 + 2hx) - x^2}{h}$$

Example 1.54

Use the above definition to find the derivative of $f(x) = x^2$

We know that

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Here $f(x+h) = (x+h)^2 = x^2 + h^2 + 2hx$ so

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{(x^2 + h^2 + 2hx) - x^2}{h}$$

$$= \lim_{h \to 0} \frac{h(h+2x)}{h} = \lim_{h \to 0} (h + 2x) = 2x$$

Exercises

Prove that

- $\frac{d}{dx}(\cos x) = -\sin x$
- $\frac{d}{dx}(e^x) = e^x$

Basic Rules of Differentiation

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(1) *Derivative of a Constant Function:* If f has the constant value $f(x) = c$, then:

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(3) *Constant Multiple Rule:* If u is a differentiable function of x, and c is a constant, then

$$\frac{d}{dx}(cu) = c\frac{du}{dx}$$

(4) *Derivative Sum Rule:* If u and v are differentiable functions of x, then their sum $u + v$ is differentiable at every point where u and v are both differentiable. At such points,

$$\frac{d}{dx}(u + v) = \frac{du}{dx} + \frac{dv}{dx}$$

(4) *Derivative Sum Rule:* If u and v are differentiable functions of x, then their sum $u+v$ is differentiable at every point where u and v are both differentiable. At such points,

$$\frac{d}{dx}(u+v) = \frac{du}{dx} + \frac{dv}{dx}$$

(5) *Derivative Product Rule:* If u and v are differentiable at x, then so is their product uv, and

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

(4) *Derivative Sum Rule:* If u and v are differentiable functions of x, then their sum $u+v$ is differentiable at every point where u and v are both differentiable. At such points,

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(5) *Derivative Product Rule:* If u and v are differentiable at x, then so is their product uv, and

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

(6) *Derivative Quotient Rule:* If u and v are differentiable at x and if $v(x) \neq 0$, then the quotient u/v is differentiable at x, and

$$\frac{d}{dx}(\frac{u}{v}) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

Example 1.55

Suppose that $f(x) = -5x^3 + 3x^2 - 9x + 7$, then find:

(a) The derivative of $f(x)$

(b) The slope of the tangent line at $x = 2$

(c) The equation of the tangent at $x = 2$

Example 1.55

Suppose that $f(x) = -5x^3 + 3x^2 - 9x + 7$, then find:

(a) The derivative of $f(x)$

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(c) The equation of the tangent at $x = 2$

Note: The equation of a line with slope m and a point (x_1, y_1) on the line is:

$$y - y_1 = m(x - x_1)$$

Example 1.55

Suppose that $f(x) = -5x^3 + 3x^2 - 9x + 7$, then find:

(a) The derivative of $f(x)$

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(a):

$$f'(x) = -15x^2 + 6x - 9$$

Example 1.55

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(a) The derivative of $f(x)$

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$$f'(x) = -15x^2 + 6x - 9$$

(b): The slope of the tangent line at $x = 2$ is $f'(2)$

Example 1.55

Suppose that $f(x) = -5x^3 + 3x^2 - 9x + 7$, then find:

(a) The derivative of $f(x)$

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Note: The equation of a line with slope m and a point (x_1, y_1) on the line is:

$$y - y_1 = m(x - x_1)$$

(a):

$$f'(x) = -15x^2 + 6x - 9$$

(b): The slope of the tangent line at $x = 2$ is $f'(2)$

$$f'(2) = -15(2)^2 + 6(2) - 9 = -15(4) + 12 - 9 = -60 + 12 - 9 = -57$$

(c): The y coordinate at $x = 2$ is

$$f(2) = -5(2)^3 + 3(2)^2 - 9(2) + 7 = -5(8) + 3(4) - 18 + 7$$

$$= -40 + 12 - 18 + 7 = -39$$

(c): The y coordinate at $x = 2$ is

$$f(2) = -5(2)^3 + 3(2)^2 - 9(2) + 7 = -5(8) + 3(4) - 18 + 7$$

$$= -40 + 12 - 18 + 7 = -39$$

So $(2, -39)$ is a point on the tangent line and the slope of the line is -57 so the equation of the line is:

$$y - y_1 = m(x - x_1) = y + 39 = -57(x - 2)$$

Example 1.56

Use the rules to show that

$$\frac{d}{dx}(\tan x) = \sec^2 x$$

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Use the rules to show that

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$$\frac{d}{dx}(\tan x) = \frac{d}{dx}(\frac{\sin x}{\cos x})$$

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Use the rules to show that

$$\frac{d}{dx}(\tan x) = \sec^2 x$$

$$\frac{d}{dx}(\tan x) = \frac{d}{dx}(\frac{\sin x}{\cos x})$$

Using the quotient rule, we will get

$$\frac{d}{dx}(\frac{\sin x}{\cos x}) = \frac{(\cos x)[\frac{d}{dx}(\sin x)] - (\sin x)[\frac{d}{dx}(\cos x)]}{(\cos^2 x)}$$

Exercises

Use the rules to show that:

- $$\frac{d}{dx}(\sec x) = \sec(x) \cdot \tan(x)$$
- $$\frac{d}{dx}(\csc(x)) = -\csc(x) \cdot \cot(x)$$

  Hint: $\csc(x) = 1/\sin(x) \quad and \quad \cot(x) = 1/\tan(x)$
- $$\frac{d}{dx}(\cot(x)) = -\csc^2(x)$$

Definition 1.57

If $f(u)$ is differentiable at the point $u = g(x)$ and $g(x)$ is differentiable at x, then the composite function $(f \circ g)(x) = f(g(x))$ is differentiable at x, and

$$(f \circ g)'(x) = f'(g(x)) \cdot g'(x)$$

In another notation, if $y = f(u)$ and $u = g(x)$, then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

Example 1.58

If $y = (x^3 + 4x^4 + 7)^{99}$, find $\frac{dy}{dx}$

Example 1.59

If $y = \dfrac{1000}{(x^4 + 2x^2 + 8)^{40}}$, find $\frac{dy}{dx}$

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Example 1.58

If $y = (x^3 + 4x^4 + 7)^{99}$, find $\frac{dy}{dx}$

Let $u = x^3 + 4x^4 + 7$, we can write y as $y = u^{99}$, then by chain rule we have:

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = 99u^{98}[3x^2 + 16x^3],$$

Example 1.59

If $y = \dfrac{1000}{(x^4 + 2x^2 + 8)^{40}}$, find $\frac{dy}{dx}$

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If $y = (x^3 + 4x^4 + 7)^{99}$, find $\frac{dy}{dx}$

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$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = 99u^{98}[3x^2 + 16x^3],$$

so

$$\frac{dy}{dx} = 99(x^3 + 4x^4 + 7)^{98}(3x^2 + 16x^3)$$

Example 1.59

If $y = \dfrac{1000}{(x^4 + 2x^2 + 8)^{40}}$, find $\frac{dy}{dx}$

$y = 1000(x^4 + 2x^2 + 8)^{-40}.$

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The Chain rule is:

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

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Let $u = x^4 + 2x^2 + 8$, so we can write $y = 1000u^{-40}$
The Chain rule is:

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

Now:

$$\frac{dy}{du} = (1000)(-40)u^{-41}$$

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Let $u = x^4 + 2x^2 + 8$, so we can write $y = 1000u^{-40}$
The Chain rule is:

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Now:

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$$\frac{du}{dx} = 4x^3 + 4x$$

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Let $u = x^4 + 2x^2 + 8$, so we can write $y = 1000u^{-40}$
The Chain rule is:

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so

$$\frac{dy}{dx} = -40000u^{-41}[4x^3 + 4x]$$

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Let $u = x^4 + 2x^2 + 8$, so we can write $y = 1000u^{-40}$
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so

$$\frac{dy}{dx} = -40000u^{-41}[4x^3 + 4x]$$

then

$$\frac{dy}{dx} = \frac{-40000}{u^{41}}(4x^3 + 4x)$$

$y = 1000(x^4 + 2x^2 + 8)^{-40}$.
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The Chain rule is:

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$$\frac{du}{dx} = 4x^3 + 4x$$

so

$$\frac{dy}{dx} = -40000u^{-41}[4x^3 + 4x]$$

then

$$\frac{dy}{dx} = \frac{-40000}{u^{41}}(4x^3 + 4x)$$

or

$$\frac{dy}{dx} = \frac{-40000(4x^3 + 4x)}{x^4 + 2x^2 + 8}$$

Note: Sometimes it is useful to involve a second (or more) intermediate function

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dv} \cdot \frac{dv}{dx}$$

so the chain rule gives

$$\frac{d\sin^4(x^5+7)}{dx} = \frac{d\sin^4(x^5+7)}{d\sin(x^5+7)} \cdot \frac{d\sin(x^5+7)}{d(x^5+7)} \cdot \frac{d(x^5+7)}{dx}$$

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$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dv} \cdot \frac{dv}{dx}$$

Example 1.60

Find $\frac{dy}{dx}$, when

$$y = \sin^4(x^5 + 7)$$

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$$\frac{d\sin^4(x^5+7)}{dx} = \frac{d\sin^4(x^5+7)}{d\sin(x^5+7)} \cdot \frac{d\sin(x^5+7)}{d(x^5+7)} \cdot \frac{d(x^5+7)}{dx}$$

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$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dv} \cdot \frac{dv}{dx}$$

Example 1.60

Find $\frac{dy}{dx}$, when

$$y = \sin^4(x^5 + 7)$$

Let $u = \sin(x^5 + 7)$ and let $v = x^5 + 7$
so the chain rule gives

$$\frac{d\sin^4(x^5+7)}{dx} = \frac{d\sin^4(x^5+7)}{d\sin(x^5+7)} \cdot \frac{d\sin(x^5+7)}{d(x^5+7)} \cdot \frac{d(x^5+7)}{dx}$$

Note: Sometimes it is useful to involve a second (or more) intermediate function

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dv} \cdot \frac{dv}{dx}$$

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Let $u = \sin(x^5 + 7)$ and let $v = x^5 + 7$
so the chain rule gives

$$\frac{d\sin^4(x^5+7)}{dx} = \frac{d\sin^4(x^5+7)}{d\sin(x^5+7)} \cdot \frac{d\sin(x^5+7)}{d(x^5+7)} \cdot \frac{d(x^5+7)}{dx}$$

$$\left[4\sin^3(x^5+7)\right] \cdot \left[\cos(x^5+7)\right] \cdot \left[5x^4\right] = 20x^4 \cdot \sin^3(x^5+7) \cdot \cos(x^5+7)$$

Exercises

Find $\frac{dy}{dx}$, if:

- $$y = x^2 e^{\sin x}$$
- $$y = \tan^3[\sin^2(x^4)]$$

Example 1.61

Find dy/dx for

$$y = \tan^3[\sin^2(x^4)]$$

Example 1.61

Find dy/dx for

$$y = \tan^3[\sin^2(x^4)]$$

$y = u^3$

Example 1.61

Find dy/dx for

$$y = \tan^3[\sin^2(x^4)]$$

$y = u^3$
$u = \tan[\sin^2(x^4)]$

Example 1.61

Find dy/dx for

$$y = \tan^3[\sin^2(x^4)]$$

$y = u^3$
$u = \tan[\sin^2(x^4)] = \tan(v)$

Example 1.61

Find dy/dx for

$$y = \tan^3[\sin^2(x^4)]$$

$y = u^3$
$u = \tan[\sin^2(x^4)] = \tan(v)$
$v = \sin^2(x^4)$

Example 1.61

Find dy/dx for

$$y = \tan^3[\sin^2(x^4)]$$

$y = u^3$
$u = \tan[\sin^2(x^4)] = \tan(v)$
$v = \sin^2(x^4) = t^2$

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Find dy/dx for

$$y = \tan^3[\sin^2(x^4)]$$

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$v = \sin^2(x^4) = t^2$
$t = \sin(x^4)$

Example 1.61

Find dy/dx for

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Example 1.61

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$t = \sin(x^4) = \sin(r)$
$r = x^4$

By the chain rule :

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dv} \cdot \frac{dv}{dt} \cdot \frac{dt}{dr} \cdot \frac{dr}{dx}$$

Example 1.61

Find dy/dx for

$$y = \tan^3[\sin^2(x^4)]$$

$y = u^3$
$u = \tan[\sin^2(x^4)] = \tan(v)$
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By the chain rule :

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dv} \cdot \frac{dv}{dt} \cdot \frac{dt}{dr} \cdot \frac{dr}{dx}$$

$$\frac{dy}{dx} = (3u^2)(sec^2(v))(2t)(\cos r)(4x)$$

Example 1.61

Find dy/dx for

$$y = \tan^3[\sin^2(x^4)]$$

$y = u^3$
$u = \tan[\sin^2(x^4)] = \tan(v)$
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$$\frac{dy}{dx} = (3u^2)(sec^2(v))(2t)(\cos r)(4x)$$

$$\frac{dy}{dx} = (3(\tan^2[\sin^2(x^4)])) \cdot (\sec^2(\sin^2(x^4))) \cdot (2\sin(x^4)) \cdot (\cos x^4) \cdot x^4$$

Differentiation of Inverse functions

It is often useful to be able to express the derivative of an inverse function in terms of the derivatives of f.

Definition 1.62

If $y = f^{-1}(x)$, then $x = f(y)$ and also

$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} = \frac{1}{f'(y)}$$

Example 1.63

Let us use the inverse rule to find $\frac{dy}{dx}$, when $y = x^{1/3}$

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Note: We know that the answer is $\frac{1}{3}x^{\frac{1}{3}-1} = \frac{1}{3}x^{\frac{-2}{3}}$, so we do not have to use the inverse rule but here we aim to use this rule to differentiate the function.

Example 1.63

Let us use the inverse rule to find $\frac{dy}{dx}$, when $y = x^{1/3}$

Note: We know that the answer is $\frac{1}{3}x^{\frac{1}{3}-1} = \frac{1}{3}x^{\frac{-2}{3}}$, so we do not have to use the inverse rule but here we aim to use this rule to differentiate the function.
If $y = x^{\frac{1}{3}}$, then $y^3 = x$, or $x = y^3$, so

$$\frac{dx}{dy} = 3y^2$$

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Let us use the inverse rule to find $\frac{dy}{dx}$, when $y = x^{1/3}$

Note: We know that the answer is $\frac{1}{3}x^{\frac{1}{3}-1} = \frac{1}{3}x^{\frac{-2}{3}}$, so we do not have to use the inverse rule but here we aim to use this rule to differentiate the function.
If $y = x^{\frac{1}{3}}$, then $y^3 = x$, or $x = y^3$, so

$$\frac{dx}{dy} = 3y^2$$

By the inverse rule:

$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} = \frac{1}{3y^2}$$

Example 1.63

Let us use the inverse rule to find $\frac{dy}{dx}$, when $y = x^{1/3}$

Note: We know that the answer is $\frac{1}{3}x^{\frac{1}{3}-1} = \frac{1}{3}x^{\frac{-2}{3}}$, so we do not have to use the inverse rule but here we aim to use this rule to differentiate the function.
If $y = x^{\frac{1}{3}}$, then $y^3 = x$, or $x = y^3$, so

$$\frac{dx}{dy} = 3y^2$$

By the inverse rule:

$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} = \frac{1}{3y^2}$$

But $y = x^{\frac{1}{3}}$ so

$$\frac{dy}{dx} = \frac{1}{3(x^{\frac{1}{3}})^2} = \frac{1}{3}x^{-\frac{2}{3}}$$

Example 1.64

Find dy/dx for

$$y = \sin^{-1} x$$

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Example 1.64

Find dy/dx for

$$y = \sin^{-1} x$$

Let $y = \sin^{-1} x$, then $x = \sin y \ \ (\star)$, so

$$\frac{dx}{dy} = \cos y \ \ (\star\star)$$

Example 1.64

Find dy/dx for

$$y = \sin^{-1} x$$

Let $y = \sin^{-1} x$, then $x = \sin y \ \ (\star)$, so

$$\frac{dx}{dy} = \cos y \ \ (\star\star)$$

Now using the inverse rule and from $(\star\star)$, we have

$$\frac{dy}{dx} = \frac{1}{\cos y} \quad or \quad \frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$$

Example 1.64

Find dy/dx for

$$y = \sin^{-1} x$$

Let $y = \sin^{-1} x$, then $x = \sin y$ $(\star)$, so

$$\frac{dx}{dy} = \cos y \quad (\star\star)$$

As $\sin^2 y + \cos^2 y = 1$, then $\cos y = \sqrt{1 - \sin^2 y}$, so by using $(\star)$

$$\cos y = \sqrt{1 - x^2}$$

Now using the inverse rule and from $(\star\star)$, we have

$$\frac{dy}{dx} = \frac{1}{\cos y} \quad or \quad \frac{dy}{dx} = \frac{1}{\sqrt{1 - x^2}}$$

Exercises

- Show that:

$$\frac{d}{dx}(\cos^{-1} x) = \frac{-1}{\sqrt{1-x^2}}$$

- Using the identity $1 + \tan^2 A = \sec^2 A$, show that:

$$\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$$

- Find

$$\frac{d}{dx}[\tan^{-1}(e^{x^2})]$$

We had

$$\frac{d}{dx}(e^x) = e^x$$

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We will find $\frac{dy}{dx}$ when $y = \ln(x)$

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If $y = \ln(x)$, then $x = e^y$, so $\frac{dx}{dy} = e^y$.

We had

$$\frac{d}{dx}(e^x) = e^x$$

We will find $\frac{dy}{dx}$ when $y = \ln(x)$
If $y = \ln(x)$, then $x = e^y$, so $\frac{dx}{dy} = e^y$.
Using the inverse rule we get

$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} = \frac{1}{e^y} = \frac{1}{x}$$

We had

$$\frac{d}{dx}(e^x) = e^x$$

We will find $\frac{dy}{dx}$ when $y = \ln(x)$
If $y = \ln(x)$, then $x = e^y$, so $\frac{dx}{dy} = e^y$.
Using the inverse rule we get

$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} = \frac{1}{e^y} = \frac{1}{x}$$

Therefore

$$\frac{d}{dx}(\ln(x)) = \frac{1}{x}$$

A function $f(x)$ can be written in a unique way as the sum of one even function and one odd function. The decomposition is

$$f(x) = \underbrace{\frac{f(x) + f(-x)}{2}}_{\text{even part}} + \underbrace{\frac{f(x) - f(-x)}{2}}_{\text{odd part}}.$$

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$$f(x) = \underbrace{\frac{f(x) + f(-x)}{2}}_{\text{even part}} + \underbrace{\frac{f(x) - f(-x)}{2}}_{\text{odd part}}.$$

$$e^x = \underbrace{\frac{e^x + e^{-x}}{2}}_{\text{even part}} + \underbrace{\frac{e^x - e^{-x}}{2}}_{\text{odd part}}.$$

Hyperbolic Functions

The hyperbolic functions are defined as follows

- *Hyperbolic sine of* x

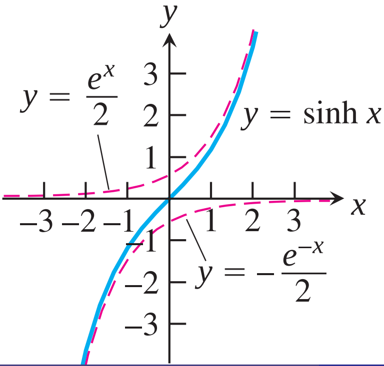
$$\sinh(x) = \frac{e^x - e^{-x}}{2}$$

Hyperbolic Functions

The hyperbolic functions are defined as follows

- *Hyperbolic sine of x*

$$\sinh(x) = \frac{e^{x} - e^{-x}}{2}$$

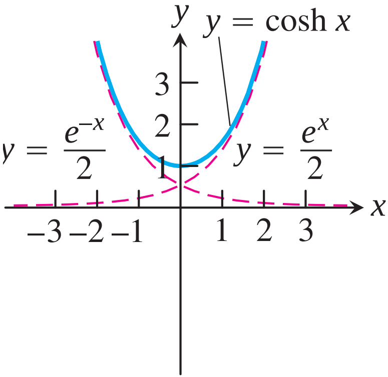


- *Hyperbolic cosine of* x

$$\cosh(x) = \frac{e^x + e^{-x}}{2}$$

- *Hyperbolic cosine of* x

$$\cosh(x) = \frac{e^x + e^{-x}}{2}$$

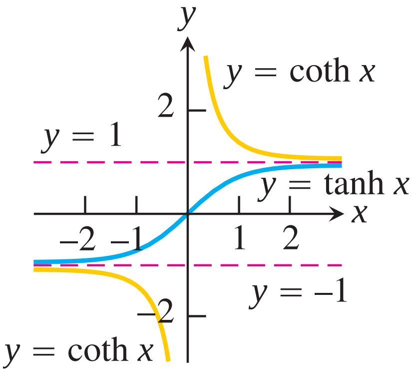


- *Hyperbolic tangent*

$$\tanh(x) = \frac{\sinh(x)}{\cosh(x)} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

- *Hyperbolic cotangent*

$$\coth(x) = \frac{\cosh(x)}{\sinh(x)} = \frac{e^x + e^{-x}}{e^x - e^{-x}}$$



Example 1.65

Find
$$\frac{d\sinh(x)}{dx}$$

Example 1.65

Find
$$\frac{d\sinh(x)}{dx}$$

We know that
$$\sinh(x) = \frac{e^x - e^{-x}}{2}$$

Example 1.65

Find

$$\frac{d\sinh(x)}{dx}$$

We know that

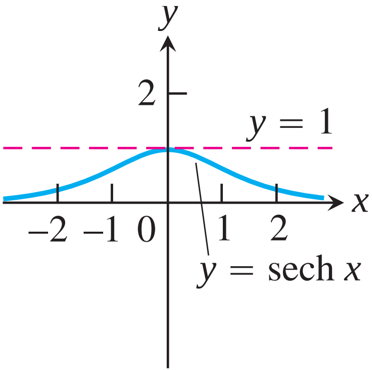
$$\sinh(x) = \frac{e^x - e^{-x}}{2}$$

So

$$\frac{d\sinh(x)}{dx} = \frac{d}{dx}(\frac{e^x - e^{-x}}{2}) = \frac{1}{2}\frac{d(e^x - e^{-x})}{dx} = \frac{1}{2}[e^x + e^{-x}] = \cosh(x)$$

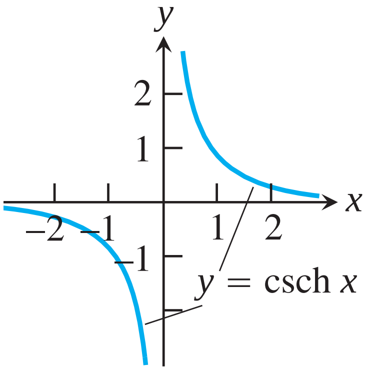
- *Hyperbolic secant*

$$\operatorname{sech}(x) = \frac{1}{\cosh(x)} = \frac{2}{e^x + e^{-x}}$$



- *Hyperbolic cosecant*

$$\operatorname{csch}(x) = \frac{1}{\sinh(x)} = \frac{2}{e^x - e^{-x}}$$



Exercises

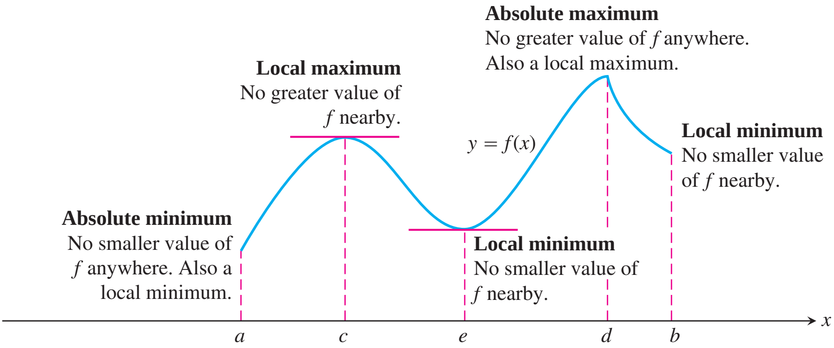
- Show that
$$\frac{d}{dx}(\cosh(x)) = \sinh(x)$$
- Show that
$$\frac{d}{dx}(\tanh(x)) = \operatorname{sech}^2(x)$$

The second derivative

The derivative of the derivative is called the second derivative.
For a function $y = f(x)$, we write the second derivative as

$$\frac{d^2y}{dx^2} \quad or \quad f''(x) \quad or \quad f^{(2)}(x)$$

The basic idea is that the optimal value of a differentiable function $f(x)$ (its maximum and minimum value) generally occurs when $f'(x) = 0$


Absolute maximum
No greater value of f anywhere.
Also a local maximum.
Local maximum
No greater value of
f nearby.
y = f(x)
Local minimum
No smaller value
of f nearby.
Absolute minimum
No smaller value of
f anywhere. Also a
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Local minimum
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x
a
c
e
d
b

Definition 1.66

An interior point of the domain of a function f where f' is zero or undefined is a *critical point* of f.

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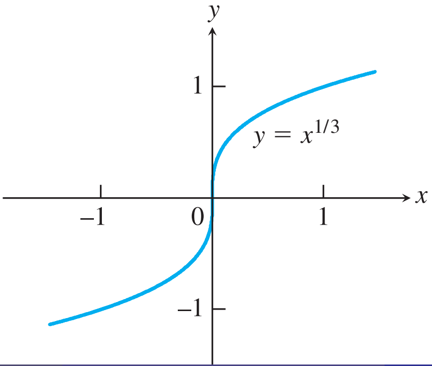
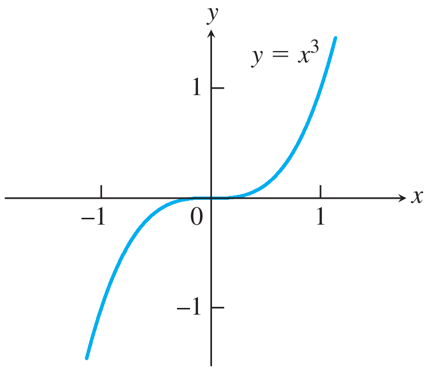


Note: Some values of x satisfying $f'(x) = 0$ are not maximum or minimum.
for example $x = 0$ is a critical point of $f(x) = x^3$, because

$$f'(x) = 3x^2 = 0 \Rightarrow x = 0$$

But $x = 0$ is neither a maximum or minimum.



The First Derivative Test

We will show how to test the critical points of a function for the presence of local maximums and minimums.

Definition 1.67

Let f be a function defined on an interval I and let x_1 and x_2 be any two points in I.

- If $f(x_1) < f(x_2)$ whenever $x_1 < x_2$, then f is said to be *increasing* on I.
- If $f(x_1) > f(x_2)$ whenever $x_1 < x_2$, then f is said to be *decreasing* on I.

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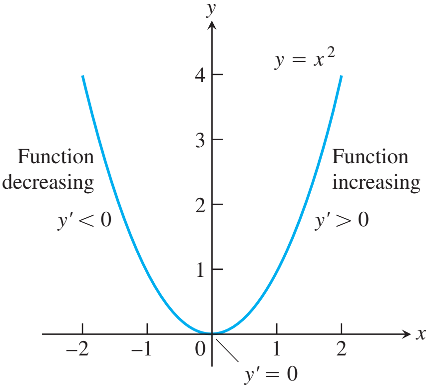
Example 1.68

The function $f(x) = x^2$ decreases on $(-\infty, 0]$ and increases on $[0, \infty)$

Theorem 1.69

Suppose that f is differentiable on (a, b)

- *If $f'(x) > 0$ at each point $x \in (a, b)$, then f is increasing*
- *If $f'(x) < 0$ at each point $x \in (a, b)$, then f is decreasing*



Example 1.70

Find the critical points of $f(x) = x^3 - 12x - 5$ and identify the intervals on which f is increasing and decreasing

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The function f is everywhere continuous and differentiable. The first derivative

$$f'(x) = 3x^2 - 12 = 3(x^2 - 4) = 3(x + 2)(x - 2)$$

is zero at $x = -2$ and $x = 2$.

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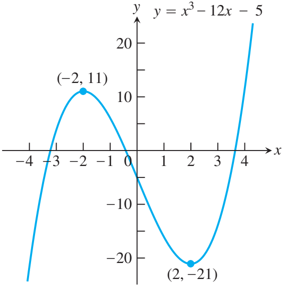
$$f'(x) = 3x^2 - 12 = 3(x^2 - 4) = 3(x + 2)(x - 2)$$

is zero at $x = -2$ and $x = 2$. These critical points subdivided the domain of f into intervals $(-\infty, -2), (-2, 2)$ and $(2, \infty)$ on which f' is either positive or negative. We determine the sign of f' by evaluating f at a convenient point in each subinterval.

		-2		2	
$3(x+2)$	$-$	$\bullet$	$+$		$+$
$x-2$	$-$		$-$	$\bullet$	$+$
$f'(x)$	$+$	$\bullet$	$-$	$\bullet$	$+$



Note: At the points where f has a minimum value, $f' < 0$ immediately to the left and $f' > 0$ immediately to the right. Thus the function is decreasing on the left of the minimum value and it is increasing on its right. Similarly, at the points where f has a maximum value, $f' > 0$ immediately to the left and $f' < 0$ immediately to the right. Thus the function is increasing on the left of the maximum value and decreasing on its right.

Theorem 1.71

First Derivative test for local maximums and minimums: Suppose that c is a critical point of f,

- *if f' changes from negative to positive at c, then f has a local minimum.*
- *if f' changes from positive to negative at c, then f has a local maximum.*
- *if f' does not change sign at c (that is, f' is positive on both sides of c or negative on both sides), then f has no maximum or minimum at c.*

Example 1.72

Find the critical points of

$$f(x) = x^{1/3}(x-4)$$

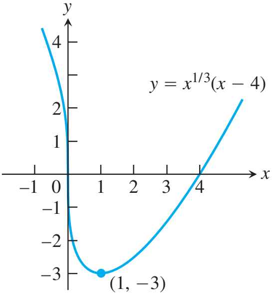
Identify the local maximums and minimums.

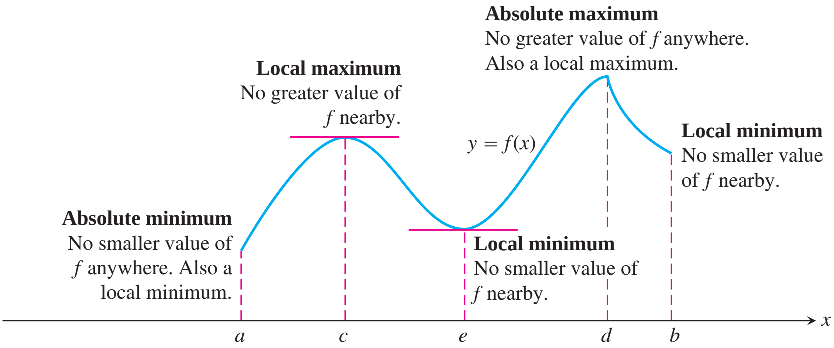
We can write $f(x) = x^{1/3}(x-4) = x^{4/3} - 4x^{1/3}$.
The first derivative

$$f'(x) = \frac{d}{dx}(x^{4/3} - 4x^{1/3}) = \frac{4}{3}x^{1/3} - \frac{4}{3}x^{-2/3} = \frac{4}{3}x^{-2/3}(x-1) = \frac{4(x-1)}{3x^{2/3}}$$

is zero at $x = 1$ and undefined at $x = 0$. So these are the critical points.

		0	1
$4(x-1)$	$-$	$-$ •	$+$
$3x^{2/3}$	$+$	$+$	$+$
$f'(x)$	$-$	$-$ •	$+$





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No smaller value of
f nearby.
x
a
c
e
d
b

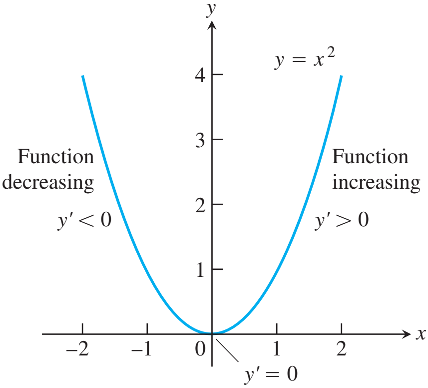
Definition 1.73

An interior point of the domain of a function f where f' is zero or undefined is a *critical point* of f.

Theorem 1.74

Suppose that f is differentiable on (a, b)

- *If $f'(x) > 0$ at each point $x \in (a, b)$, then f is increasing*
- *If $f'(x) < 0$ at each point $x \in (a, b)$, then f is decreasing*



Theorem 1.75

First Derivative test for local maximums and minimums: Suppose that c is a critical point of f,

- *if f' changes from negative to positive at c, then f has a local minimum.*
- *if f' changes from positive to negative at c, then f has a local maximum.*
- *if f' does not change sign at c (that is, f' is positive on both sides of c or negative on both sides), then f has no maximum or minimum at c.*

Definition 1.76

The graph of a differentiable function $y = f(x)$ is:

- **concave up** on an open interval I if f' is increasing on I
- **concave down** on an open interval I if f' is decreasing on I

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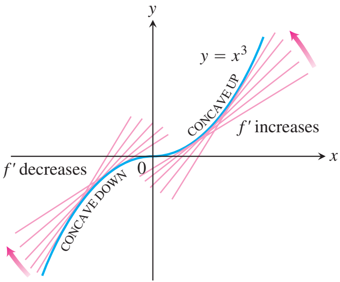
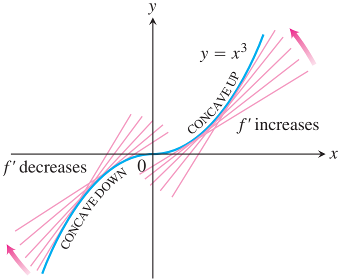
- **concave down** on an open interval I if f' is decreasing on I

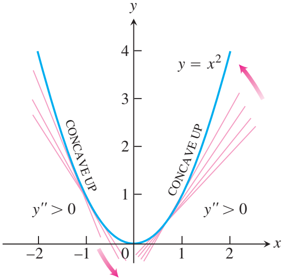
Theorem 1.77

Let $y = f(x)$ be twice-differentiable on an open interval I.

- *If $f'' > 0$ on I, the graph of f over I is concave up*
- *If $f'' < 0$ on I, the graph of f over I is concave down.*


y
$y = x^3$
CONCAVE UP
f' increases
x
f' decreases
0
CONCAVE DOWN


y
x
$y = x^3$
CONCAVE UP
f' increases
f' decreases
0
CONCAVE DOWN


y
x
4
3
2
1
$y = x^2$
CONCAVE UP
CONCAVE UP
$y'' > 0$
$y'' > 0$
−2
−1
0
1
2

Definition 1.78

A point where the graph of a function has a tangent line and where the concavity changes is a **point of inflection**

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Note: To find the points of inflection, we need to find the zeros of f'' and points where f'' is not defined. Then we need to check the concavity.

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Example 1.79

The curve $y = x^4$ has no inflection point at $x = 0$. Even though $y'' = 12x^2$ is zero there, it does not change sign

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Example 1.79

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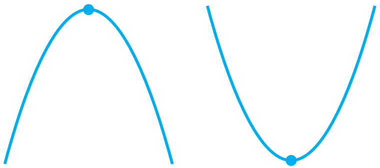
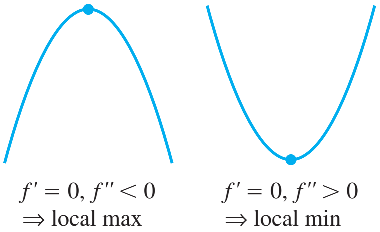
Note: An inflection point may not exist where $y'' = 0$



$f' = 0, f'' < 0$
$\Rightarrow$ local max

$f' = 0, f'' > 0$
$\Rightarrow$ local min



Theorem 1.80

Second derivative test for local maximums and minimums: suppose that f'' is continuous on an open interval that contains $x = c$.

- *If $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum at $x = c$*
- *If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at $x = c$*
- *If $f'(c) = 0$ and $f''(c) = 0$, then the test fails. The function f may have a local maximum, a local minimum, or neither.*

Example 1.81

Find and classify the stationary points and the points of inflection of

$$f(x) = 4x^3 - 21x^2 + 18x + 6$$

$f'(x) = 12x^2 - 42x + 18$
When $f'(x) = 0$, we have:

$$12x^2 - 42x + 18 = 0 \Rightarrow 2x^2 - 7x + 3 = 0 \Rightarrow (2x - 1)(x - 3) = 0$$

So The stationary points are: $x = 1/2$ and $x = 3$
$f''(x) = 24x - 42$ so

$$f''(1/2) = 24(1/2) - 42 = 12 - 42 < 0$$

which means $x = 1/2$ is a local max. Also as

$$f''(3) = 24(3) - 42 = 72 - 42 > 0$$

$x = 3$ is a local min.

Example 1.82

sketch a graph of the function $f(x) = x^4 - 4x^3 + 10$

We use the following steps:

(1) Find the critical (stationary) points
(2) Find the points of inflection
(3) Use the second derivative test
(4) Find the y-value of these points to pair them
(5) Draw the table to find the intervals on which f is increasing and the intervals on which f is decreasing
(6) Add some extra rows to your table to see where the graph of f is concave up and where it is concave down
(7) Plot some specific points, such as local maximum and minimum points, points of inflection, and intercepts.
(8) Sketch the general shape of the graph for f

Step (1):

$$f'(x) = 4x^3 - 12x^2 = 4x^2(x-3) = 0$$

So the stationary points are $x = 0$ and $x = 3$

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$$f'(x) = 4x^3 - 12x^2 = 4x^2(x-3) = 0$$

So the stationary points are $x = 0$ and $x = 3$

Step (2):

$$f''(x) = 12x^2 - 24x = 12x(x-2) = 0$$

So the points of inflection are $x = 0$ and $x = 2$

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$$f'(x) = 4x^3 - 12x^2 = 4x^2(x-3) = 0$$

So the stationary points are $x = 0$ and $x = 3$
Step (2):

$$f''(x) = 12x^2 - 24x = 12x(x-2) = 0$$

So the points of inflection are $x = 0$ and $x = 2$
Step (3):
At $x = 0$, $f''(0) = 0$ so the test fails in this point. But at $x = 3$, $f''(3) = 36 > 0$ so based on the test, $x = 3$ is a local minimum

Step (1):

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So the points of inflection are $x = 0$ and $x = 2$

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At $x = 0$, $f''(0) = 0$ so the test fails in this point. But at $x = 3$, $f''(3) = 36 > 0$ so based on the test, $x = 3$ is a local minimum

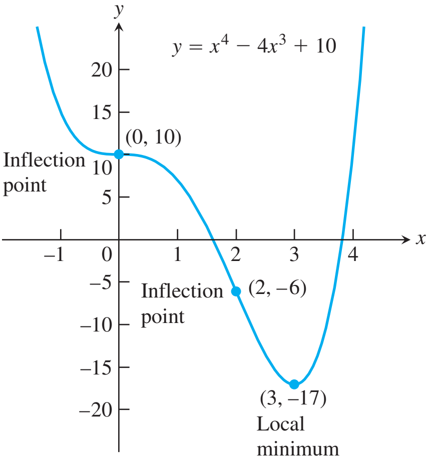
Step 4:

$f(0) = 10$, $f(2) = -6$ and $f(3) = -17$

Step 5 and 6:

		0		2		3	
$4x^2$	$+$	$\bullet$	$+$		$+$		$+$
$x-3$	$-$		$-$		$-$	$\bullet$	$+$
$f'(x)$	$-$	$\bullet$	$-$		$-$	$\bullet$	$+$
$12x$	$-$	$\bullet$	$+$		$+$		$+$
$x-2$	$-$		$-$	$\bullet$	$+$		$+$
$f''(x)$	$+$	$\bullet$	$-$	$\bullet$	$+$		$+$

Step 7 and 8:


y
$y = x^4 - 4x^3 + 10$
20
15
10
5
–5
–10
–15
–20
–1
0
1
2
3
4
x
(0, 10)
Inflection
point
Inflection
point
(2, –6)
(3, –17)
Local
minimum

Example 1.83

Sketch the graph of

$$f(x) = \frac{(x+1)^2}{1+x^2}$$

Step (1):

$f(x) = \dfrac{(x+1)^2}{1+x^2}$, so

$$f'(x) = \frac{(1+x^2)\cdot 2(x+1) - (x+1)^2 \cdot 2x}{(1+x^2)^2} = \frac{2(1-x^2)}{(1+x^2)^2}$$

So the critical points are $x = -1$ and $x = 1$.

Step (2):

$$f''(x) = \frac{(1+x^2)^2 \cdot 2(-2x) - 2(1-x^2)[2(1+x^2)\cdot 2x]}{(1+x^2)^4} = \frac{4x(x^2-3)}{(1+x^2)^3}$$

So inflection points are $x = \sqrt{3}$, $x = 0$ and $x = -\sqrt{3}$

Step (3):

since $f''(1) = -1 < 0$ so $x = 1$ is a local max and since $f''(-1) = 1 > 0$ so $x = -1$ is a local min

Step (4):

$f(1) = 2$, $f(-1) = 0$, $f(\sqrt{3}) = \frac{(\sqrt{3}+1)^2}{4}$, $f(-\sqrt{3}) = \frac{(-\sqrt{3}+1)^2}{4}$, $f(0) = 1$

We can also find the asymptotes of the function:

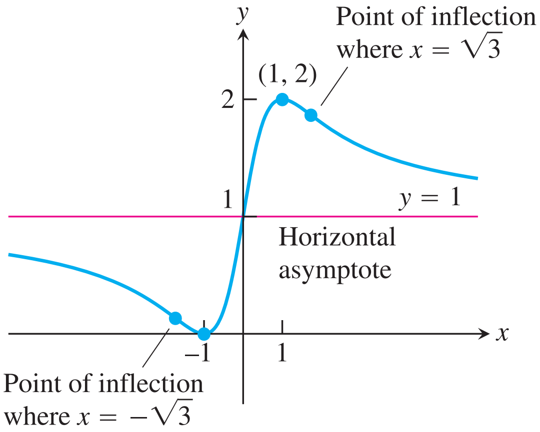
$$\lim_{x \to \infty} f(x) = 1$$

So $y = 1$ is the horizontal asymptote.

Step 5 and 6:

		$-\sqrt{3}$		-1		0		1		$\sqrt{3}$	
$2(1-x^2)$	$-$		$-$	•	$+$		$+$	•	$-$		$-$
$(1+x^2)^2$	$+$		$+$		$+$		$+$		$+$		$+$
$f'(x)$	$-$		$-$	•	$+$		$+$	•	$-$		$-$
$4x$	$-$		$-$		$-$	•	$+$		$+$		$+$
(x^2-3)	$+$	•	$-$		$-$		$-$		$-$	•	$+$
$(1+x^2)^3$	$+$		$+$		$+$		$+$		$+$		$+$
$f''(x)$	$-$	•	$+$		$+$	•	$-$		$-$	•	$+$

Step 7 and 8:


y
Point of inflection
where $x = \sqrt{3}$
(1, 2)
2
1
$y = 1$
Horizontal
asymptote
x
−1
1
Point of inflection
where $x = -\sqrt{3}$

Exercise

Use the steps of the graphing procedure to graph the following equations:

- $$y = x^2 - 4x + 3$$
- $$y = x^3 - 3x + 3$$
- $$y = \frac{x^2 - 3}{x - 2}$$

Example 1.84

Sketch the graph of the function

$$f(x) = \frac{1}{1 + e^{3x}}$$

Example 1.84

Sketch the graph of the function

$$f(x) = \frac{1}{1+e^{3x}}$$

Step (1):

Find the critical points, we can write $f(x)$ as, $f(x) = (1+e^{3x})^{-1}$, so

$$f'(x) = -3(e^{3x})(1+e^{3x})^{-2} = \frac{-3e^{3x}}{(1+e^{3x})^2}$$

As for all $x \in \mathbb{R}$, $e^{3x} > 0$, so $f'(x)$ has no root and it is always negative which means that the function is always decreasing.

Example 1.84

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As for all $x \in \mathbb{R}$, $e^{3x} > 0$, so $f'(x)$ has no root and it is always negative which means that the function is always decreasing.

Step (2):

Find the points of inflection:

$$f''(x) = \frac{(1 + e^{3x})^2[-9e^{3x}] - [-3e^{3x}][6e^{3x}(1 + e^{3x})]}{(1 + e^{3x})^4}$$

Setting $f''(x) = 0$, we have

$$(1 + e^{3x})^2(-9e^{3x}) + [3e^{3x}][6e^{3x}(1 + e^{3x})] = 0$$

$$\Rightarrow (1 + e^{3x})(-9) + 18e^{3x} = 0 \Rightarrow 1 + e^{3x} = 2e^{3x} \Rightarrow 1 = e^{3x}$$

So $x = 0$ could be the point of inflection, we need to check whether the second derivative changes sign around this point or not.

Setting $f''(x) = 0$, we have

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Step (3):

No need to use the second derivative test, as there is no critical point.

Setting $f''(x) = 0$, we have

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So $x = 0$ could be the point of inflection, we need to check whether the second derivative changes sign around this point or not.

Step (3):

No need to use the second derivative test, as there is no critical point.

Step (4):

$f(0) = 1/2$, the function has no x-intercept as $f(x)$ is never zero.

As $f(x)$ is a rational function. we should also find the asymptotes

$$\lim_{x\to\infty} \frac{1}{1+e^{3x}} = 0$$

also

$$\lim_{x\to-\infty} \frac{1}{1+e^{3x}} = \frac{1}{1+0} = 1$$

Is the denominator equal to zero? No, because there is no real x when $1+e^{3x}=0$

So the asymptotes are:

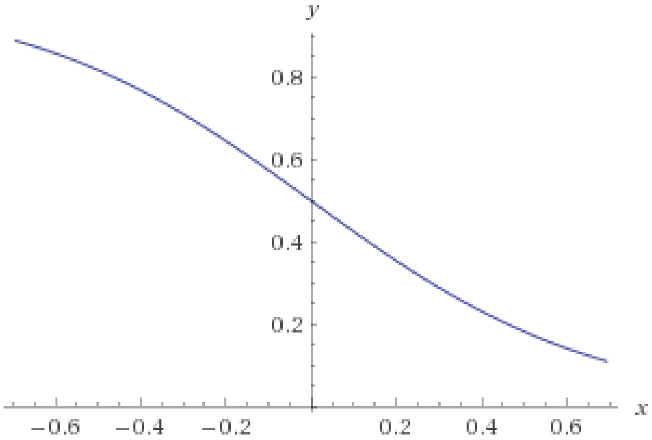
$y=0$ when $x\to\infty$

$y=1$ when $x\to-\infty$

Step 5 and 6: $f'(x) = \dfrac{-3e^{3x}}{(1+e^{3x})^2}$ and $f''(x) = \dfrac{9(e^{3x}-1)}{(1+e^{3x})^4}$

Step 5 and 6: $f'(x) = \dfrac{-3e^{3x}}{(1+e^{3x})^2}$ and $f''(x) = \dfrac{9(e^{3x}-1)}{(1+e^{3x})^4}$

		0	
$f'(x)$	$-$		$-$
$f''(x)$	$-$	$\bullet$	$+$


y
0.8
0.6
0.4
0.2
x
−0.6
−0.4
−0.2
0.2
0.4
0.6

Logarithmic Differentiation

One could use the product and quotient rule to differentiate the following type of function.

Example 1.85

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First note:

$$\frac{d}{dx}(\ln f(x)) = \frac{d\ln f(x)}{df(x)} \cdot \frac{df(x)}{dx} = \frac{1}{f(x)} \cdot \frac{df(x)}{dx} = \frac{1}{f(x)} \cdot f'(x)$$

So differentiating the above:

$$\frac{f'(x)}{f(x)} = \frac{2x}{x^2+5} + \frac{2}{x+3} + \frac{1}{2(x+7)} - \frac{5}{x+4}$$

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So

$$f'(x) = \left[\frac{(x^2+5)(x+3)^2\sqrt{x+7}}{(x+4)^5}\right] \cdot \left[\frac{2x}{x^2+5} + \frac{2}{x+3} + \frac{1}{2(x+7)} - \frac{5}{x+4}\right]$$

Continuous and Differentiable Function

Example 1.86

Determine values of k and m such that $h(x)$ is continuous and differentiable at all points.

$$h(x) = \begin{cases} 4x + k & if \quad x < 3 \\ mx^3 - 1 & if \quad x \geq 3 \end{cases}$$

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So,

$$4(3) + k = m(3)^3 - 1 \Rightarrow 12 + k = 27m - 1 \Rightarrow k = 27m - 13$$

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Also

$$\lim_{x \to 3^+} \frac{(mx^3 - 1) - (27m - 1)}{x - 3} = \lim_{x \to 3^+} \frac{m(x^3 - 27)}{x - 3}$$

$$= \lim_{x \to 3^+} m(x^2 + 3x + 9) = 27m$$

so

$$4 = 3m(3^2) \Rightarrow 4 = 27m \Rightarrow m = 27/4$$

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As $k = 27m - 13$, $k = 27(4/27) - 13 \Rightarrow k = 4 - 13 = -9$

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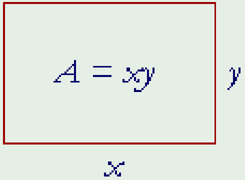
(4) *Write an equation for the unknown quantity.* If you can, express the unknown as a function of a single variable or in two equations in two unknowns. This may require considerable manipulation.

Example 1.87

A rectangle has the following side lengths



Find x and y if the area is to be maximized if the perimeter equals 30m.

The perimeter equals $30m$, it means $2(x+y)=30$ or $x+y=15$, so

$$y = 15 - x, (\star)$$

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Let:

$$\frac{dA}{dx} = 0 \Rightarrow x = \frac{15}{2} (y = \frac{15}{2})$$

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Let:

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So $x = \frac{15}{2}$ is a critical point.

Now we use the second derivative test

$$\frac{d^2A}{dx^2} = 2 < 0$$

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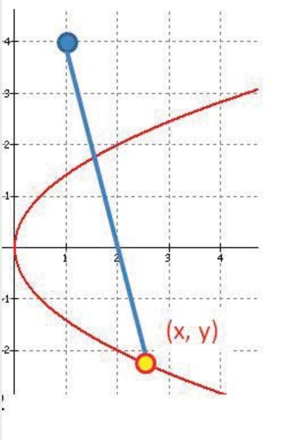
Example 1.88

Find the point on the parabola $y^2 = 2x$, closest to the point $(1, 4)$



The distance between the two points (x, y) and $(1, 4)$ equals:

$$d = \sqrt{(x-1)^2 + (y-4)^2}$$

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As the coordinates of the point satisfy $y^2 = 2x$ or $\frac{y^2}{2} = x$, let $y = t$ then $x = \frac{t^2}{2}$. So $(x, y) = (\frac{t^2}{2}, t)$ and $(1, 4)$ have distance squared equal to

$$r(t) = (\frac{t^2}{2} - 1)^2 + (t-4)^2$$

$$r(t) = \frac{t^4}{4} - t^2 + 1 + t^2 + 16 - 8t = \frac{t^4}{4} - 8t + 17$$

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So $t = 2$ is the only critical point as the other factor does not have any real roots

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$$d''(t) = 3t^2 \Rightarrow d''(2) = 12 > 0$$

So $t = 2$ is a local minimum, therefore $(2, 2)$ is the point on $y^2 = 2x$ closest to $(1, 4)$

suppose that $f(a) = g(a) = 0$, that $f'(a)$ and $g'(a)$ exist, and $g'(a) \neq 0$. Then:

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{f'(a)}{g'(a)}$$

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Proof: Working backward from $f'(a)$ and $g'(a)$, which are themselves limits, we have

$$\frac{f'(a)}{g'(a)} = \frac{\lim_{x \to a} \frac{f(x)-f(a)}{x-a}}{\lim_{x \to a} \frac{g(x)-g(a)}{x-a}} = \lim_{x \to a} \frac{\frac{f(x)-f(a)}{x-a}}{\frac{g(x)-g(a)}{x-a}}$$

$$= \lim_{x \to a} \frac{f(x) - f(a)}{g(x) - g(a)} = \lim_{x \to a} \frac{f(x) - 0}{g(x) - 0} = \lim_{x \to a} \frac{f(x)}{g(x)}$$

Theorem 1.89

L'Hôpital's Rule:

If f *and* g *are differentiable and* $g'(x) \neq 0$ *near* a *(except possibly at* a*),*

- *if* $\lim_{x\to a} f(x) = 0$ *and* $\lim_{x\to a} g(x) = 0$
  or
- $\lim_{x\to a} f(x) = \pm\infty$ *and* $\lim_{x\to a} g(x) = \pm\infty$

Then:

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

Example 1.90

Find the following limit:

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As $\lim_{x\to 1}(\ln x) = \ln(1) = 0$ and $\lim_{x\to 1}(x-1) = 0$, we can apply L'Hôpital's Rule:

$$\lim_{x\to 1} \frac{\ln x}{x-1} \overset{\text{H}}{=} \lim_{x\to 1} \frac{\frac{d}{dx}(\ln x)}{\frac{d}{dx}(x-1)} = \lim_{x\to 1} \frac{\frac{1}{x}}{1} = \lim_{x\to 1} \frac{1}{x} = \frac{1}{1} = 1$$

Example 1.91

Find

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$\lim_{x\to\infty} e^x = \infty$ and $\lim_{x\to\infty} x^2 = \infty$, so we can apply the L'Hôpital's Rule:

$$\lim_{x\to\infty} \frac{e^x}{x^2} \overset{\text{H}}{=} \lim_{x\to\infty} \frac{e^x}{2x}$$

This result is also of the form $\frac{\infty}{\infty}$, so we can apply the L'Hôpital's Rule again:

$$\lim_{x\to\infty} \frac{e^x}{2x} \overset{\text{H}}{=} \lim_{x\to\infty} \frac{e^x}{2} = \infty$$

Example 1.92

Find

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$\lim_{x\to 0}(\sin(x) - x) = 0$ and $\lim_{x\to 0} x^3 = 0$, as we see the limit is of the form $\frac{0}{0}$. so

$$\lim_{x\to 0} \frac{\sin(x) - x}{x^3} \overset{\text{H}}{=} \lim_{x\to 0} \frac{\cos(x) - 1}{3x^2}$$

Note that $\lim_{x\to 0}(\cos x - 1) = 0$ and $\lim_{x\to 0}(3x^2) = 0$, so again we get $\frac{0}{0}$, so we can to apply L'Hôpital's Rule again,

$$\lim_{x\to 0} \frac{\cos(x) - 1}{3x^2} \overset{\text{H}}{=} \lim_{x\to 0} \frac{-\sin x}{6x} = \frac{0}{0}$$

We can use L'Hôpital's Rule again:

$$\lim_{x\to 0} \frac{-\sin x}{6x} \overset{\text{H}}{=} \lim_{x\to 0} \frac{-\cos x}{6} = \frac{-1}{6}$$

Definition 2.1

A function F is an *antiderivative* of f on an interval I if $F'(x) = f(x)$ for all x in I.

f is the derivative of $F \longleftrightarrow F$ is an antiderivative of f

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Where c is an arbitrary constant.

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For example the antiderivative of $f(x) = 2x$ equals $F(x) = x^2 + c$
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For example the antiderivative of $f(x) = 3x^2$ equals $F(x) = x^3 + c$

Definition 2.2

We call

$$\int f(x)dx$$

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Example 2.4

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Example 2.4

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  Because the derivative of $\frac{1}{2}x^2 + c$ is equal to x
- $$\int x^2 dx = \frac{1}{3}x^3 + c$$

In general when $x \neq -1$:

$$\int x^n dx = \frac{x^{n+1}}{n+1} + c$$

Example 2.5

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$$= [2(\frac{x^3}{3}) + c_1] + [9(\frac{x^8}{8}) + c_2] = 2(\frac{x^3}{3}) + 9(\frac{x^8}{8}) + (c_1 + c_2)$$

$$\int (2x^2 + 9x^7)dx = 2(\frac{x^3}{3}) + 9(\frac{x^8}{8}) + c$$

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$$\int (2x^2 + 9x^7)dx = 2(\frac{x^3}{3}) + 9(\frac{x^8}{8}) + c$$

Theorem 2.7

If $u = g(x)$ is a differentiable function, then:

$$\int f(g(x))g'(x)dx = \int f(u)du$$

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Replace u by x^3, we get:

$$\int 3x^2 \sin(x^3) dx = \int \sin(u) du = \int \sin(u) du = -\cos(u) + c = -\cos(x^3) + c$$

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$$\int 2x\sqrt{1+x^2}dx$$

We define a function of x, called u.

Let $u = 1 + x^2$, so $\dfrac{du}{dx} = 2x$ or $du = 2xdx$ So the above integral becomes:

$$\int \sqrt{u}du = \int u^{\frac{1}{2}}du = \frac{u^{\frac{3}{2}}}{\frac{3}{2}} + c = \frac{2}{3}u^{\frac{3}{2}} + c = \frac{2}{3}(1+x^2)^{\frac{3}{2}} + c$$

Example 2.10

Find:

$$\int \cos(4x - 7)dx$$

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Let $u = 4x - 7$.

Example 2.10

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then the above integral equals:

$$\frac{1}{4}\int \cos(u)du = \frac{1}{4}\sin(u) + c = \frac{1}{4}\sin(4x - 7) + c$$

Example 2.11

Find:

$$\int e^{-5x} dx$$

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Let $u = -5x$,

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Find:

$$\int e^{-5x}dx$$

Let $u = -5x$, then $\dfrac{du}{dx} = -5$ or $du = -5dx$.

Example 2.11

Find:

$$\int e^{-5x}dx$$

Let $u = -5x$, then $\dfrac{du}{dx} = -5$ or $du = -5dx$.
We can rewrite the original integral as:

$$\frac{-1}{5}\int -5e^{-5x}dx$$

then the above integral equals:

$$\frac{-1}{5}\int e^{u}du = \frac{-1}{5}e^{u} + c = \frac{-1}{5}e^{-5x} + c$$

Example 2.12

Find:

$$\int \sin^3 x \cos x dx$$

Definition 2.13

Rational functions have the general form

$$f(x) = \frac{p(x)}{q(x)}$$

where $p(x)$ and $q(x)$ are polynomials.

- **IF** degree of $p(x) <$ degree of $q(x)$, then $f(x)$ is a strictly proper rational function.
- **IF** degree of $p(x) =$ degree of $q(x)$, then $f(x)$ is a proper rational function.
- **IF** degree of $p(x) >$ degree of $q(x)$, then $f(x)$ is an improper rational function.

An improper or proper rational function can be expressed in terms of a strictly proper rational function

Example 2.14

Express $f(x) = \dfrac{3x^4 + 2x^3 - 5x^2 + 6x - 7}{x^2 - 2x + 3}$ in terms of a strictly proper rational function

$$f(x) = 3x^2 + 8x + 2 - \frac{14x + 13}{x^2 - 2x + 3}$$

(1) Linear factor to the power of 1:
A linear factor $(x-a)$ gives rise to the partial fraction of the form

$$\frac{A}{x-a}$$

(2) Linear factor to the power of greater than 1:
If $(x-\alpha)^k$ appears in the denominator, it will give rise to the following terms:

$$\frac{A_1}{x-\alpha}+\frac{A_2}{(x-\alpha)^2}+\cdots+\frac{A_k}{(x-\alpha)^k}$$

(3) Irreducible quadratic factors:
An irreducible quadratic $ax^2 + bx + c$ gives rise to partial fractions of the form
$$\frac{Ax + B}{ax^2 + bx + c}$$

(4) Irreducible quadratic factors to the power of greater than 1:
If $(ax^2 + bx + c)^k$ appears in the denominator, it will give rise to the following terms:

$$\frac{A_1x + B_1}{ax^2 + bx + c} + \frac{A_2x + B_2}{(ax^2 + bx + c)^2} + \cdots + \frac{A_kx + B_k}{(ax^2 + bx + c)^k}$$

Example 2.15

Evaluate

$$\int \frac{3x+4}{x^2+7x+12}dx$$

Example 2.15

Evaluate

$$\int \frac{3x+4}{x^2+7x+12}dx$$

$x^2 + 7x + 12 = (x + 4)(x + 3)$,

Example 2.15

Evaluate

$$\int \frac{3x+4}{x^2+7x+12}dx$$

$x^2+7x+12 = (x+4)(x+3)$, so the partial fraction decomposition has the form:

$$\frac{3x+4}{x^2+7x+12} = \frac{A}{x+4} + \frac{B}{x+3}$$

Example 2.15

Evaluate
$$\int \frac{3x+4}{x^2+7x+12}dx$$

$x^2+7x+12=(x+4)(x+3)$, so the partial fraction decomposition has the form:
$$\frac{3x+4}{x^2+7x+12}=\frac{A}{x+4}+\frac{B}{x+3}$$

To find the values of the undetermined coefficients A and B, we add the new fractions:

$$\frac{3x+4}{x^2+7x+12}=\frac{A}{x+4}+\frac{B}{x+3}=\frac{A(x+3)+B(x+4)}{(x+4)(x+3)}$$

The denominators on both sides of the above equation are identical, so the numerators must be equal:

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$$3x+4 = A(x+3)+B(x+4) = Ax+3A+Bx+4B = (A+B)x+(3A+4B)$$

which means:

$$\begin{cases} A + B &= 3 \\ 3A + 4B &= 4 \end{cases}$$

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which means:

$$\begin{cases} A+B &= 3 \\ 3A+4B &= 4 \end{cases}$$

So $A = 3 - B$ so inserting this into the second equation, we get:

$$3(3-B)+4B = 4 \Rightarrow 9-9B+4B = 4 \Rightarrow 9-5B = 4 \Rightarrow 5B = 5 \Rightarrow B = 1$$

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Therefore $A = 3 - B = 3 - 1 = 2$.
So

$$\frac{3x+4}{x^2+7x+12} = \frac{2}{x+4} + \frac{1}{x+3}$$

Now we can express the integral as:

$$\int \frac{3x+4}{x^2+7x+12}dx = \int \Big(\frac{2}{x+4} + \frac{1}{x+3}\Big)dx$$

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$$\int \frac{3x+4}{x^2+7x+12}dx = \int \Big(\frac{2}{x+4} + \frac{1}{x+3}\Big)dx$$

Using the sum rule, we have:

$$\int \frac{3x+4}{x^2+7x+12}dx = \int \Big(\frac{2}{x+4} + \frac{1}{x+3}\Big)dx = \underbrace{\int \frac{2}{x+4}dx}_{I_1} + \underbrace{\int \frac{1}{x+3}dx}_{I_2}$$

First we find I_1:

$$I_1 = \int \frac{2}{x+4}dx = 2\int \frac{1}{x+4}dx$$

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Let $u = x + 4$, then $du = dx$,

First we find I_1:

$$I_1 = \int \frac{2}{x+4}dx = 2\int \frac{1}{x+4}dx$$

Let $u = x+4$, then $du = dx$, so:

$$I_1 = \int \frac{2}{x+4}dx = 2\int \frac{1}{x+4}dx = 2\int \frac{du}{u} = 2\ln(u)+c_1 = 2\ln(x+4)+c_1$$

First we find I_1:

$$I_1 = \int \frac{2}{x+4}dx = 2\int \frac{1}{x+4}dx$$

Let $u = x + 4,$ then $du = dx,$ so:

$$I_1 = \int \frac{2}{x+4}dx = 2\int \frac{1}{x+4}dx = 2\int \frac{du}{u} = 2\ln(u)+c_1 = 2\ln(x+4)+c_1$$

Therefore $I_1 = 2\ln(x+4) + c_1$

$$I_2 = \int \frac{1}{x+3} dx$$

Let $u = x + 3$, then $du = dx$, so:

$$I_2 = \int \frac{1}{x+3} dx = \int \frac{du}{u} = \ln(u) + c_2 = \ln(x+3) + c_2$$

Therefore $I_2 = \ln(x+3) + c_2$

$$I_2 = \int \frac{1}{x+3} dx$$

Let $u = x + 3$, then $du = dx$, so:

$$I_2 = \int \frac{1}{x+3} dx = \int \frac{du}{u} = \ln(u) + c_2 = \ln(x+3) + c_2$$

Therefore $I_2 = \ln(x+3) + c_2$
So finally

$$\int \frac{3x+4}{x^2+7x+12} dx = \int \Big(\frac{2}{x+4} + \frac{1}{x+3}\Big) dx = \underbrace{\int \frac{2}{x+4} dx}_{I_1} + \underbrace{\int \frac{1}{x+3} dx}_{I_2}$$

$$= 2\ln(x+4) + c_1 + I_2 = \ln(x+3) + c_2 = 2\ln(x+4) + \ln(x+3) + c$$

Example 2.16

Evaluate:

$$\int \frac{dx}{x(x^2+1)^2}$$

The form of the partial fraction decomposition is

$$\frac{1}{x(x^2+1)^2} = \frac{A}{x} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{(x^2+1)^2}$$

Add up the fractions:

$$\frac{1}{x(x^2+1)^2} = \frac{A}{x} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{(x^2+1)^2}$$

$$= \frac{A(x^2+1)^2 + (Bx+C)x(x^2+1) + (Dx+E)x}{x(x^2+1)^2}$$

So:

$$1 = A(x^2+1)^2 + (Bx+C)x(x^2+1) + (Dx+E)x$$
$$\Rightarrow 1 = A(x^4+2x^2+1) + B(x^4+x^2) + C(x^3+x) + Dx^2 + Ex$$
$$\Rightarrow 1 = (A+B)x^4 + Cx^3 + (2A+B+D)x^2 + (C+E)x + A$$

If we equate coefficients, we get the system

$$\begin{cases} A+B &= 0 \\ C &= 0 \\ 2A+B+D &= 0 \\ C+E &= 0 \\ A &= 1 \end{cases}$$

Solving this system gives: $A = 1, B = -1, C = 0, D = -1, E = 0$
Thus

$$\int \frac{dx}{x(x^2+1)^2} = \int \Big(\frac{A}{x} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{(x^2+1)^2}\Big)dx$$

$$= \int \Big(\frac{1}{x} + \frac{-x+0}{x^2+1} + \frac{-x+0}{(x^2+1)^2}\Big)dx$$

$$= \int \Big(\frac{1}{x} + \frac{-x}{x^2+1} + \frac{-x}{(x^2+1)^2}\Big)dx$$

$$= \underbrace{\int \frac{1}{x}dx}_{I_1} + \underbrace{\int \frac{-x}{x^2+1}dx}_{I_2} + \underbrace{\int \frac{-x}{(x^2+1)^2}dx}_{I_3}$$

$$I_1 = \int \frac{1}{x} dx$$

Therefore

$$I_1 = \ln(x) + c_1$$

Also

$$I_2 = \int \frac{-x}{x^2+1} dx$$

Let $u = x^2 + 1$ then $du = 2xdx$, so $-\frac{du}{2} = -xdx$

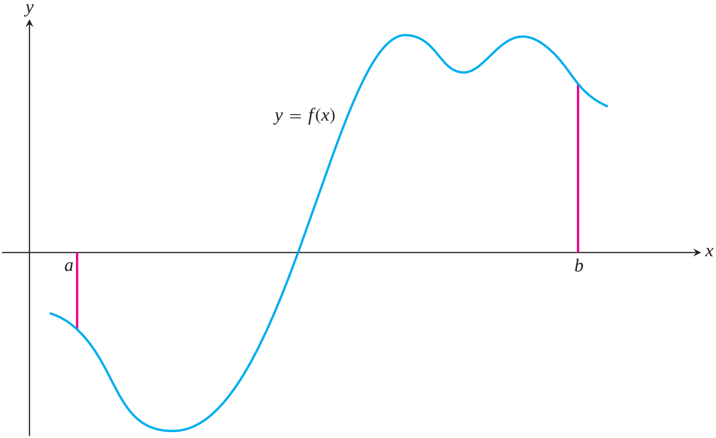
$$\int \frac{-x}{x^2+1} dx = \int \frac{-du}{2u} = \frac{-1}{2} \int \frac{du}{u} = \frac{-1}{2} \ln u + c_2 = \frac{-1}{2} \ln(x^2+1) + c_2$$

$$I_3 = \int \frac{-x}{(x^2+1)^2} dx$$

Let $u = x^2 + 1$, then $du = 2xdx$, so $\frac{-du}{2} = -xdx$, Therefore:

$$I_3 = \int \frac{-x}{(x^2+1)^2} dx = \int \frac{-du}{2(u)^2} = \frac{-1}{2} \int \frac{du}{u^2} = \frac{-1}{2} \int u^{-2} du$$

$$= \frac{-1}{2}\Big(\frac{u^{-2+1}}{-2+1}\Big) + c_3 = \frac{-1}{2}\Big(\frac{(x^2+1)^{-2+1}}{-2+1}\Big) + c_3 = \frac{-1}{2}\Big(\frac{(x^2+1)^{-1}}{-1}\Big) + c_3$$



y
x
a
b
y = f(x)

To do so, we choose $n-1$ points $\{x_1, x_2, \cdots, x_{n-1}\}$ between a and b and satisfying

$$a < x_1 < x_2 < \cdots < x_{n-1} < b$$

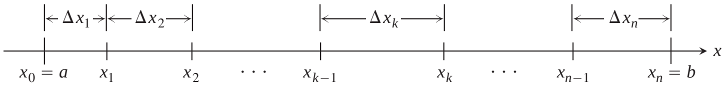
To make the notation consistent, we denote a by x_0 and b by x_n, so that :

$$a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b$$

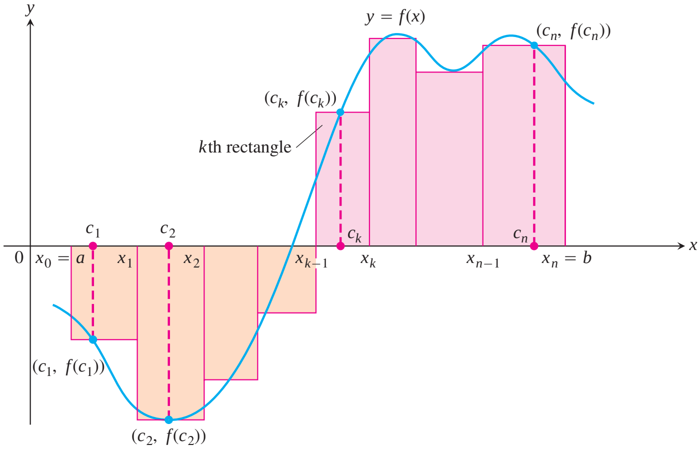
The set

$$P = \{x_0, x_1, x_2, \cdots, x_{n-1}, x_n\}$$

is called a partition of $[a, b]$

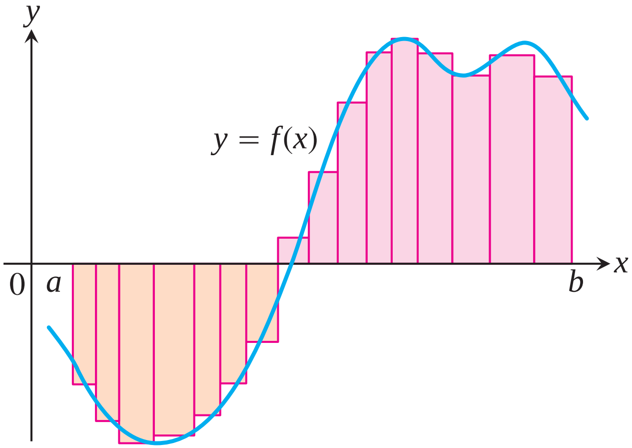


In each subinterval we select some point c_k. Then on each subinterval we stand a vertical rectangle that stretches from the x-axis to touch the curve at $(c_k, f(c_k))$:



We can also let all the the subintervals to have equal widths, $\Delta x_k = \frac{1}{n}$
Then the area of each rectangle is

$$\Delta x_k \cdot f(c_k) = \frac{1}{n} \cdot f(c_k)$$

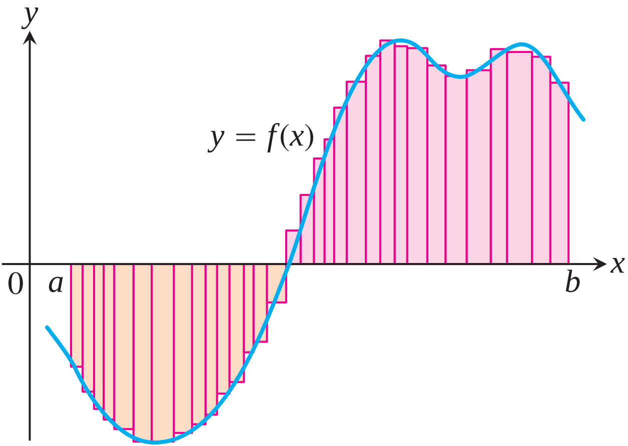


Finally we sum all these products to get

$$S_P = \sum_{k=1}^{n} \frac{1}{n} \cdot f(c_k)$$

S_P is called a Riemann sum for f on the interval $[a, b]$.

We can also make the width of the subintervals smaller.



If we make n larger (or make $\frac{1}{n}$ smaller), then

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{n}\cdot f(c_k)$$

will give us an accurate answer.
We say that the definite integral of f from a to b is:

$$\int_a^b f(x)dx = \lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{n}\cdot f(c_k)$$

The function is the integrand.

Upper limit of integration

x is the variable of integration.

Integral sign

$$\underbrace{\int_a^b f(x)\, dx}_{\text{Integral of } f \text{ from } a \text{ to } b}$$

Lower limit of integration

When you find the value of the integral, you have evaluated the integral.

Theorem 2.17

The Fundamental Theorem of Calculus (1):
Suppose that f is a continuous function on $[a, b]$, and F is a anti-derivative of f. Then:

$$\int_a^b f(x)dx = F(x)\Big]_a^b = F(b) - F(a)$$

Example 2.18

Evaluate

$$\int_0^1 x^3 dx$$

An anti-derivative can be obtained by evaluating the following integral:

$$\int x^3 dx = x^4/4$$

So $F(x) = x^4/4$, therefore:

$$\int_0^1 x^3 dx = x^4/4\Big]_0^1 = (1^4/4) - (0^4/4) = 1/4$$

Theorem 2.19

- *Order of integration:*

$$\int_a^b f(x)dx = -\int_b^a f(x)dx$$

- *Zero width interval*

$$\int_a^a f(x)dx = 0$$

- *if $a < c < b$, then*

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

Example 2.20

Suppose that $\int_{-1}^{1} f(x)dx = 5$, $\int_{1}^{4} f(x)dx = -2$ and $\int_{-1}^{1} h(x)dx = 7$. Then find

- $$\int_{4}^{1} f(x)dx$$
- $$\int_{-1}^{1} [2f(x) + 3h(x)]dx$$
- $$\int_{-1}^{4} f(x)dx$$

Theorem 2.21

The Fundamental Theorem of Calculus (2):

Suppose that f is a continuous function on $[a, b]$,

$$\frac{d}{dx}\int_a^x f(t)dt = f(x)$$

Example 2.22

Find

$$\frac{d}{dx}\int_0^x \sqrt{1+2t}dt$$

Recall that

$$\frac{d}{dx}\int_a^x f(t)dt = f(x)$$

so:

$$\frac{d}{dx}\int_0^x \sqrt{1+2t}dt = \sqrt{1+2x}$$

Example 2.23

Find $\frac{dy}{dx}$, if

$$y = \int_a^x \cos(t)dt$$

By The fundamental theorem of calculus, we know that:

$$\frac{d}{dx}\int_a^x f(t)dt = f(x)$$

So:

$$\frac{d}{dx}\int_a^x \cos(t)dt = \cos(x)$$

Example 2.24

Find $\frac{dy}{dx}$, if

$$y = \int_1^{x^2} \cos(t)dt$$

The upper limit of integration is not x but x^2. This makes y a composite of the two functions,

$$y = \int_1^{u} \cos(t)dt \quad and \quad u = x^2$$

We must therefore apply the chain rule when finding $\frac{dy}{dx}$.

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \left(\frac{d}{du}\int_1^{u} \cos(t)dt\right) \cdot \frac{du}{dx}$$

$$= \cos(u) \cdot \frac{du}{dx} = \cos(x^2) \cdot 2x = 2x \cdot \cos(x^2)$$

Example 2.25

Find $\frac{dy}{dx}$, if

$$y = \int_3^{\sqrt{x}} \frac{\cos(t)}{t} dt$$

By the chain rule:

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \left(\frac{d}{du} \int_3^{u} \frac{\cos(t)}{t} dt \right) \cdot \frac{du}{dx}$$

$$= \frac{\cos(u)}{u} \cdot \frac{du}{dx} = \frac{\cos(\sqrt{x})}{\sqrt{x}} \cdot \frac{1}{2\sqrt{x}}$$

Integration by parts is a technique for simplifying integrals of the form:

$$\int f(x) \cdot g(x) dx$$

It is useful when f can be differentiated repeatedly and g can be integrated repeatedly without difficulty. The integral

$$\int xe^x dx$$

is such an integral because $f(x) = x$ can be differentiated twice to become zero and $g(x) = e^x$ can be integrated repeatedly without difficulty.

Theorem 2.26

Integration by Parts Formula:

$$\int u dv = uv - \int v du$$

Theorem 2.26

Integration by Parts Formula:

$$\int u dv = uv - \int v du$$

Example 2.27

Find

$$\int x \cos(x) dx$$

Theorem 2.26

Integration by Parts Formula:

$$\int u dv = uv - \int v du$$

Example 2.27

Find

$$\int x\cos(x)dx$$

We use the formula $\int u dv = uv - \int v du$ with

$$u = x, \quad dv = \cos(x)dx,$$

$$du = dx, \quad v = \sin(x)$$

Then

$$\int x\cos(x)dx = x\sin(x) - \int \sin(x)dx = x\sin(x) + \cos(x) + c$$

Example 2.28

Find

$$\int \ln(x)dx$$

Since

$$\int \ln(x)dx$$

can be written as

$$\int \ln(x) \cdot 1dx$$

, we use integration by parts:

$$u = \ln(x) \quad dv = dx$$

$$du = \frac{1}{x}dx, \quad v = x$$

Then:

$$\int \ln(x)dx = x\ln(x) - \int dx = x\ln(x) - x + c$$

Example 2.29

Evaluate

$$\int x^2 e^x dx$$

With

$$u = x^2, \quad dv = e^x dx$$

$$du = 2x dx, \quad v = e^x$$

Using integration by parts,

$$\int u dv = uv - \int v du$$

We get:

$$\int x^2 e^x dx = x^2 e^x - 2 \int x e^x dx$$

$$\int xe^x dx = ?$$

we have to use integration by parts more than once. We integrate by parts again with

$$u = x, \quad dv = e^x dx$$

$$du = dx, \quad v = e^x$$

So

$$\int xe^x dx = xe^x - \int e^x dx = xe^x - e^x + c$$

Hence:

$$\int x^2 e^x dx = x^2 e^x - 2\int xe^x dx = x^2 e^x - 2xe^x + 2e^x + c$$

Example 2.30

Evaluate:

$$\int e^x \cos(x)dx$$

Let

$$u = e^x, \quad dv = \cos(x)dx$$

Then:

$$du = e^x dx, \quad v = \sin(x)$$

Using integration by parts,

$$\int udv = uv - \int vdu$$

We get:

$$\int e^x \cos(x)dx = e^x \sin(x) - \int e^x \sin(x)dx$$

The second integral is like the first except that it has $\sin x$ in place of $\cos x$. To evaluate it, we use integration by parts with:

$$u = e^x, \quad dv = \sin(x)dx$$

$$v = -\cos(x), \quad du = e^x dx$$

Then:

$$\int e^x \cos(x)dx = e^x \sin(x) - \Big(- e^x \cos(x) - \int (-\cos(x))(e^x dx) \Big)$$

$$= e^x \sin(x) + e^x \cos(x) - \int e^x \cos(x)dx$$

The unknown integral now appears on both sides of the equation. Adding the integral to both sides and adding the constant of integration gives:

$$2\int e^x\cos(x)dx = e^x\sin(x) + e^x\cos(x) + c$$

Dividing by 2 and renaming the constant of integration gives:

$$\int e^x\cos(x)dx = \frac{e^x\sin(x) + e^x\cos(x)}{2} + c$$

Infinite Limits of Integration:

Example 2.31

Evaluate:

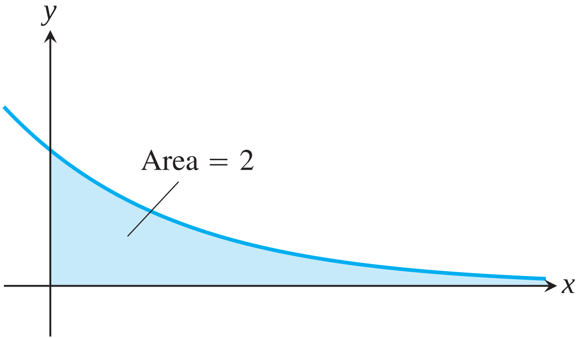
$$\int_0^\infty \frac{1}{e^{\frac{x}{2}}}dx$$

Infinite Limits of Integration:

Example 2.31

Evaluate:

$$\int_0^{\infty} \frac{1}{e^{\frac{x}{2}}}\,dx$$



First find the area $A(b)$ of the portion of the region that is bounded on the right by $x = b$.

$$A(b) = \int_0^b \frac{1}{e^{\frac{x}{2}}} dx = \int_0^b e^{-x/2} dx = -2e^{-x/2}\Big]_0^b = -2e^{-b/2} + 2$$

Then find the limit of $A(b)$ as $b \to \infty$:

$$\lim_{b\to\infty} A(b) = \lim_{b\to\infty} (-2e^{-b/2} + 2) = 2$$

Inverse Trigonometric Functions:

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- $$\int \frac{dx}{\sqrt{x^2-1}} = \sec^{-1}(x) + c$$

Example 2.32

Evaluate

$$I = \int \frac{2}{3 + 4x^2} dx$$

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$$I = \int \frac{2}{3+4x^2}dx$$

Recall that

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then $du = \frac{2}{\sqrt{3}}dx$,

Example 2.32

Evaluate

$$I = \int \frac{2}{3+4x^2}dx$$

Recall that

$$\int \frac{dx}{1+x^2} = \tan^{-1}(x) + c$$

$$I = \int \frac{2}{3(1+\frac{4}{3}x^2)}dx = \frac{2}{3}\int \frac{1}{1+\frac{4}{3}x^2}dx = \frac{2}{3}\int \frac{1}{1+(\frac{2x}{\sqrt{3}})^2}dx$$

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Example 2.32

Evaluate

$$I = \int \frac{2}{3 + 4x^2} dx$$

Recall that

$$\int \frac{dx}{1 + x^2} = \tan^{-1}(x) + c$$

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Let $u = \frac{2}{\sqrt{3}}x$, then $du = \frac{2}{\sqrt{3}}dx$, so $\frac{\sqrt{3}}{2}du = dx$.

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$$\int \frac{dx}{1+x^2} = \tan^{-1}(x) + c$$

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Let $u = \frac{2}{\sqrt{3}}x$, then $du = \frac{2}{\sqrt{3}}dx$, so $\frac{\sqrt{3}}{2}du = dx$.

$$I = \frac{2}{3}\int \frac{1}{1+u^2}(\frac{\sqrt{3}}{2}du)$$

$$I = \frac{2}{3}\frac{\sqrt{3}}{2}\int \frac{1}{1+u^2}du$$

So

$$I = \frac{1}{\sqrt{3}}\tan^{-1} u + c = \frac{1}{\sqrt{3}}\tan^{-1}(\frac{2}{\sqrt{3}}x) + c$$

To compute the area of the region bounded by the graph of a function $y = f(x)$ and the $x-$axis requires more care when the function takes on both positive and negative values. We must be careful to break up the interval $[a, b]$ into subintervals on which the function doesnt change sign. Otherwise we might get cancellation between positive and negative signed areas, leading to an incorrect total.

Definition 2.33

To find the area between the graph of $y = f(x)$ and the $x-$axis over the interval $[a, b]$, do the following:

(1) Subdivide $[a, b]$ at the zeros of f

(2) Integrate f over each subinterval.

(3) Add the absolute values of the integrals.

Example 2.34

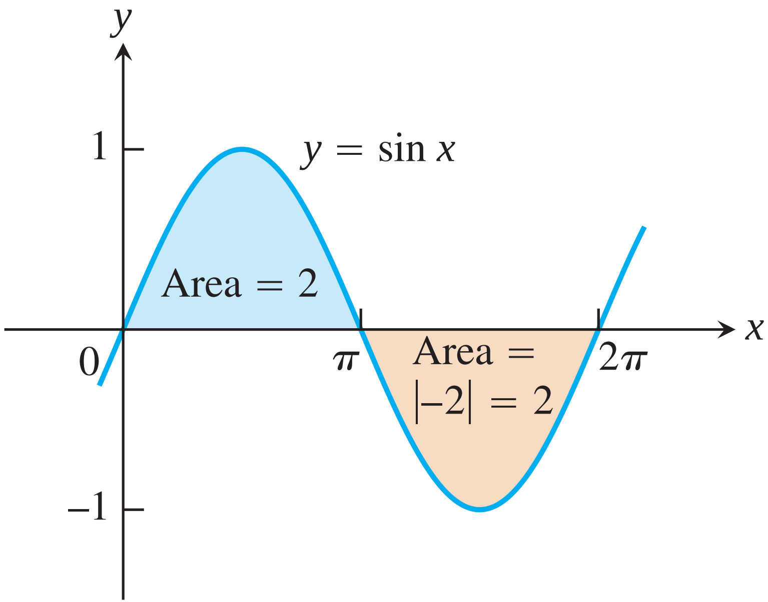
Suppose that $f(x) = \sin(x)$, find:

- $$\int_0^{2\pi} \sin(x)dx$$
- The area between the graph of $f(x)$ and the $x-$axis over $[0, 2\pi]$

The definite integral for $f(x) = \sin(x)$ is given by

$$\int_0^{2\pi} \sin(x)dx = -\cos(x)\Big]_0^{2\pi} = -[\cos(2\pi) - \cos(0)] = -[1-1] = 0$$

The definite integral is zero because the portions of the graph above and below the x-axis make canceling contributions.



y
1
y = sin x
Area = 2
x
0
π
Area =
|−2| = 2
2π
−1

The area between the graph of $f(x)$ and the $x-$axis over $[0, 2\pi]$ is calculated by breaking up the domain of $\sin(x)$ into two pieces: the interval $[0, \pi]$ over which it is nonnegative and the interval $[\pi, 2\pi]$ over which it is nonpositive.

$$\int_0^{\pi} \sin(x)dx = -\cos(x)\Big]_0^{\pi} = -[\cos(\pi) - \cos(0)] = -[-1 - 1] = 2$$

$$\int_{\pi}^{2\pi} \sin(x)dx = -\cos(x)\Big]_{\pi}^{2\pi} = -[\cos(2\pi) - \cos(\pi)] = -[1 - (-1)] = -2$$

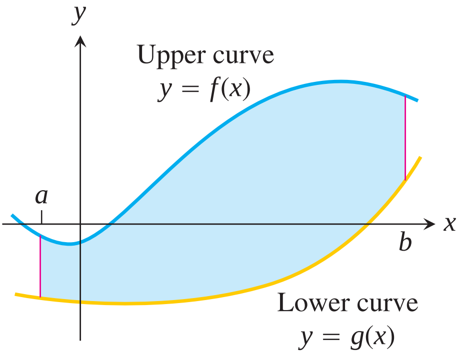
The second integral gives a negative value. The area between the graph and the axis is obtained by adding the absolute values

$$area = |2| + |-2| = 4$$

Definition 2.35

If f and g are continuous with $f(x) \geq g(x)$ throughout $[a, b]$, then the area of the region between the curves $y = f(x)$ and $y = g(x)$ from a to b is the integral of $(f - g)$ from a to b:

$$A = \int_a^b [f(x) - g(x)]dx$$



Example 2.36

Find the area of the region enclosed by the parabola $y = 2 - x^2$ and the line $y = -x$

First we sketch the two curves

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Find the area of the region enclosed by the parabola $y = 2 - x^2$ and the line $y = -x$

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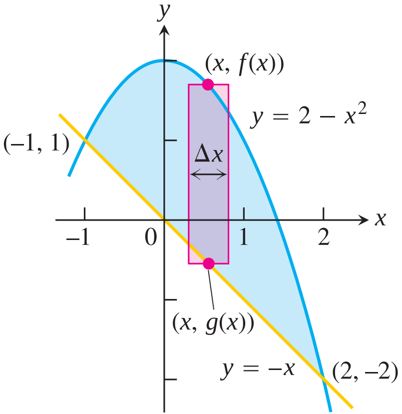


The limits of integration are found by solving $y = 2 - x^2$ and $y = -x$ simultaneously for x.

$$2 - x^2 = -x \Rightarrow x^2 - x - 2 = 0 \Rightarrow (x+1)(x-2) = 0 \Rightarrow x = -1, x = 2$$

The region runs from $x = -1$ to $x = 2$. The limits of integration are $a = -1$, $b = 2$. The area between the curve is

$$A = \int_a^b [f(x) - g(x)]dx = \int_{-1}^2 [(2 - x^2) - (-x)]dx$$

$$= \int_{-1}^2 (2 + x - x^2)dx = \left[2x + \frac{x^2}{2} - \frac{x^3}{3}\right]_{-1}^2$$

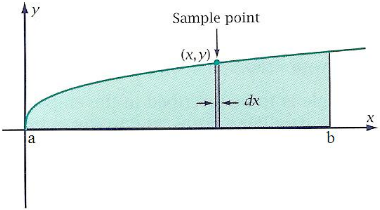
$$= \left(4 + \frac{4}{2} - \frac{8}{3}\right) - \left(-2 + \frac{1}{2} + \frac{1}{3}\right) = \frac{9}{2}$$

Example 2.37

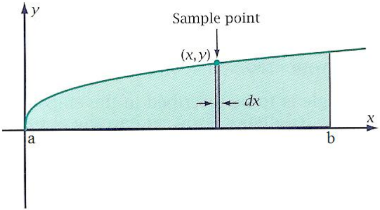
Find the area enclosed between the two curves $f(x) = 6 - 2x^2$ and $g(x) = 4x$.

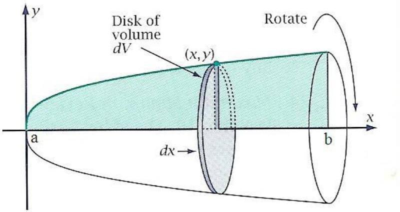
Find the volume of solids:

Find the volume of solids:



Find the volume of solids:





The solid generated by rotating a plane region about an axis in its plane is called a solid of revolution. To find the volume of a solid we need only observe that the cross-sectional area $A(x)$ is the area of a disk of radius $R(x)$, the distance of the planar regions boundary from the axis of revolution. The area is then

$$A(x) = \pi(radius)^2 = \pi[R(x)]^2$$

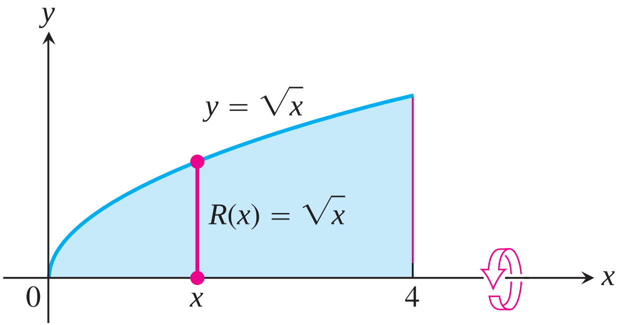
So the definition of volume gives:

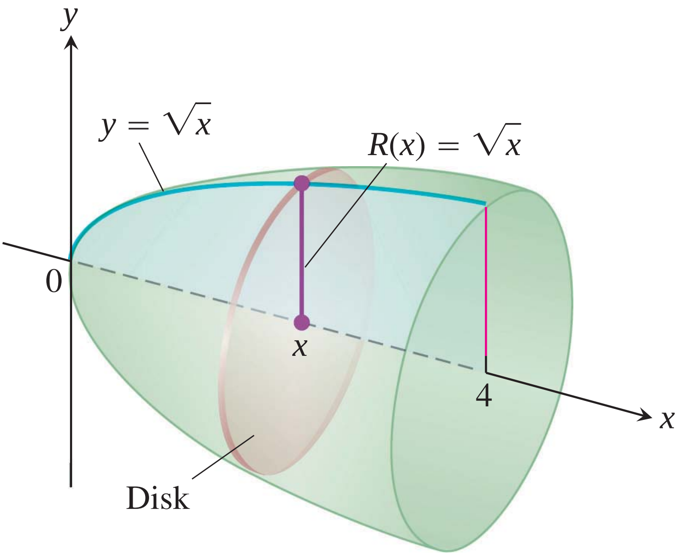
$$V = \int_a^b A(x)dx = \int_a^b \pi[R(x)]^2 dx$$

Example 2.38

The region between the curve $y = \sqrt{x}$, $0 \leq x \leq 4$, and the $x-$axis is revolved about the $x-$axis to generate a solid. Find its volume.

We draw figures showing the region, a typical radius, and the generated solid.




y
$y = \sqrt{x}$
$R(x) = \sqrt{x}$
0
x
4
x
Disk

The volume is:

$$V = \int_a^b \pi[R(x)]^2 dx == \int_0^4 \pi[\sqrt{x}]^2 dx$$

$$= \pi \int_0^4 x dx = \pi \frac{x^2}{2}\Big]_0^4 = \pi \frac{(4)^2}{2} = 8\pi$$

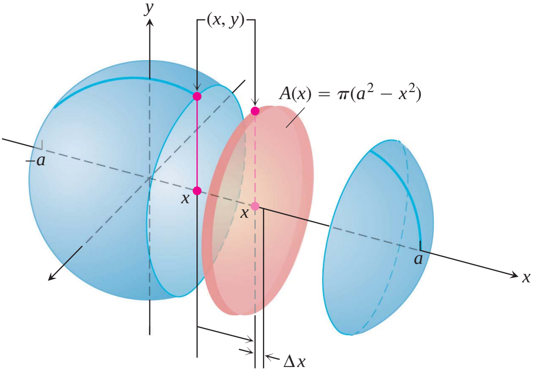
Example 2.39

The circle

$$x^2 + y^2 = a^2$$

is rotated about the $x-$axis to generate a sphere. find its volume

We imagine the sphere cut into thin slices,



the cross-sectional area at a typical point x between $-a$ and a is:

$$A(x) = \pi y^2 = \pi(a^2 - x^2)$$

Therefore, the volume is

$$V = \int_{-a}^{a} A(x)dx = \int_{-a}^{a} \pi(a^2 - x^2)dx$$

$$= \pi\left[a^2x - \frac{x^3}{3}\right]_{-a}^{a} = \frac{4}{3}\pi a^3$$

Example 2.40

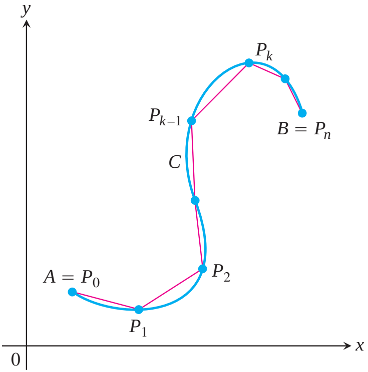
Sketch the area between $x = 2$ and $x = 3$, under the curve

$$y = \frac{1}{x-1}$$

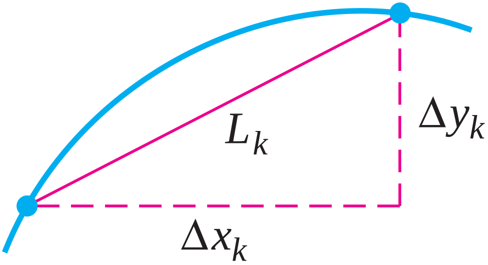
Also find the volume of the solid if this area is rotated around the $x-$axis.

Arc Length:

Arc Length: Let C be a curve given by the equation $y = f(x)$. It may be helpful to imagine the curve as the path of a particle moving from point A to pint B. We subdivide the path (or arc) AB into n pieces at points $A = P_0, P_1, P_2, \cdots, P_n = B$. Join successive points of this subdivision by straight line segments



A representative line segment:



has length:

$$L_k = \sqrt{(\Delta x_k)^2 + (\Delta y_k)^2}$$

An intuitive approximation to the length of the curve AB, S, is the sum of all the lengths L_k :

$$S \cong \sum_{k=1}^{n} L_k = \sum_{k=1}^{n} \sqrt{(\Delta x_k)^2 + (\Delta y_k)^2}$$

So

$$S \cong \sum_{k=1}^{n} \sqrt{(\Delta x_k)^2 (1 + (\frac{\Delta y_k^2}{\Delta x_k^2}))}$$

So:

$$S \cong \sum_{k=1}^{n} \sqrt{(1 + (\frac{\Delta y_k^2}{\Delta x_k^2}))}(\Delta x_k)$$

Using a Riemann sum approach.
Let $\Delta x \to 0$ or $n \to \infty$, we get:

$$S = \lim_{n\to\infty} \sum_{k=1}^{n} \sqrt{(1 + (\frac{\Delta y_k^2}{\Delta x_k^2}))}(\Delta x_k)$$

So:

$$S = \int_a^b \sqrt{1 + (\frac{dy}{dx})^2}dx$$

Theorem 2.41

If f is continuously differentiable on the closed interval $[a, b]$, the length of the curve (graph) $y = f(x)$ from $x = a$ to $x = b$ is:

$$S = \int_a^b \sqrt{1 + (\frac{dy}{dx})^2} dx = \int_a^b \sqrt{1 + (f'(x))^2} dx$$

Example 2.42

Find the length of the curve

$$y = \frac{4\sqrt{2}}{3}x^{3/2} - 1, \quad 0 \le x \le 1$$

We use the theorem with $a = 0$, $b = 1$, and

$$y = \frac{4\sqrt{2}}{3}x^{3/2} - 1$$

First we take the derivative of y:

$$\frac{dy}{dx} = \frac{4\sqrt{2}}{3} \cdot \frac{3}{2}x^{1/2} = 2\sqrt{2}x^{1/2}$$

So

$$(\frac{dy}{dx})^2 = (2\sqrt{2}x^{1/2})^2 = 8x$$

The length of the curve from $x = 0$ to $x = 1$ is:

$$S = \int_0^1 \sqrt{1 + (\frac{dy}{dx})^2} dx = \int_0^1 \sqrt{1 + 8x} dx$$

$$= \frac{2}{3} \cdot \frac{1}{8} (1 + 8x)^{3/2} \Big]_0^1 = \frac{13}{6}$$

Example 2.43

The cable of a suspension bridge takes the shape of the curve:

$$y = \frac{h}{l^2}x^2 - \frac{2h}{l}x + h$$

Where $0 \leq x \leq 2l$, $h > 0$. Find the length of the cable.

$$\frac{dy}{dx} = \frac{2h}{l^2}x - \frac{2h}{l} = \frac{2h}{l}(\frac{x}{l} - 1)$$

So:

$$(\frac{dy}{dx})^2 = \left[\frac{2h}{l}(\frac{x}{l} - 1)\right]^2$$

The length of the curve equals:

$$S = \int_a^b \sqrt{1 + (\frac{dy}{dx})^2} dx = \int_0^{2l} \sqrt{1 + \left[\frac{2h}{l}(\frac{x}{l} - 1)\right]^2} dx$$

First we find the following indefinite integral:

$$\int \sqrt{1 + \left[\frac{2h}{l}(\frac{x}{l} - 1)\right]^2} dx, \quad (\star)$$

Let $u = \frac{2h}{l}(\frac{x}{l} - 1)$, then $du = \frac{2h}{l^2}dx$ or $\frac{l^2}{2h}du = dx$
So:

$$(\star) = \frac{l^2}{2h}\int \sqrt{1+u^2}du$$

quick reminder:

- $\sinh(x) = \frac{e^x - e^{-x}}{2}$
- $\cosh(x) = \frac{e^x + e^{-x}}{2}$
- $\frac{d}{dx}\sinh(x) = \cosh(x)$
- $\frac{d}{dx}\cosh(x) = \sinh(x)$
- $\cosh^2(x) - \sinh^2(x) = 1$

Now let $u = \sinh(v)$, so $du = \cosh(v)dv$, so:

$$(\color{red}{\star}) = \frac{l^2}{2h}\int \sqrt{1+u^2}du = \frac{l^2}{2h}\int \sqrt{1+\sinh^2(v)}\cosh(v)dv$$

$$= \frac{l^2}{2h}\int \cosh(v)\cosh(v)dv = \frac{l^2}{2h}\int \cosh^2(v)dv$$

We have:

$$\cosh^2(v) = \frac{[e^{2v}+e^{-2v}+2]}{4} = \frac{1}{2}[\frac{e^{2v}+e^{-2v}+2}{2}]$$

$$= \frac{1}{2}[\frac{e^{2v}+e^{-2v}}{2}+1] = \frac{1}{2}[\cosh(2v)+1]$$

So the above integral is:

$$(\color{red}{\star}) = \frac{l^2}{2h}\int \frac{1}{2}[\cosh(2v)+1]dv = \frac{l^2}{4h}\left[\frac{1}{2}\sinh(2v)+v\right]+c$$

So:

$$(\star) = \frac{l^2}{4h}\Big[\sinh(v)\sqrt{1+\sinh^2(v)}+v\Big]+c = \frac{l^2}{4h}\Big[u\sqrt{1+u^2}+\sinh^{-1}(u)\Big]+c$$

$$= \frac{l^2}{4h}\Big[\frac{2h}{l}(\frac{x}{l}-1)\sqrt{1+(\frac{2h}{l}(\frac{x}{l}-1))^2}+\sinh^{-1}(\frac{2h}{l}(\frac{x}{l}-1))\Big]+c$$

So the length of the cable is:

$$\frac{l^2}{4h}\Big[\frac{2h}{l}(\frac{x}{l}-1)\sqrt{1+(\frac{2h}{l}(\frac{x}{l}-1))^2}+\sinh^{-1}(\frac{2h}{l}(\frac{x}{l}-1))\Big]+c\Big]_0^{2l}$$

$$= \Big(\frac{l^2}{4h}(\frac{2h}{l}\sqrt{1+(\frac{2h}{l})^2}+\sinh^{-1}(\frac{2h}{l})\Big)$$

$$-\Big(\frac{l^2}{4h}(\frac{-2h}{l}\sqrt{1+(\frac{2h}{l})^2}+\sinh^{-1}(\frac{-2h}{l})\Big)$$

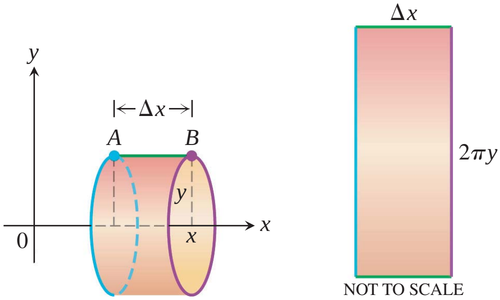
Using the fact that $\sinh^{-1}(-x) = -\sinh^{-1}(x)$, we can simplify the above answer:

$$= \sqrt{l^2 + 4h^2} + \frac{l^2}{2h}\sinh^{-1}(\frac{2h}{l})$$

We begin by rotating the line segment $y = r$, from $x = A$ to $x = B$.
If we rotate this line segment AB having length Δx about the $x-$axis, we generate a cylinder with surface area

$$2\pi y \Delta x = 2\pi r \Delta x$$

This area is the same as that of a rectangle with side lengths Δx and $2\pi y$



Surface area:

Surface area: We are interested in the surface generated by rotating the curve about the x-axis.

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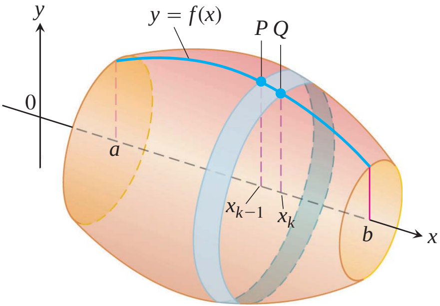


Suppose that the arc length from p to Q is ΔS_k, so the surface area of the typical band above is:

$$\Delta S_k 2\pi f(x_k)$$

So the surface area can be approximated by the following sum:

$$S \cong \sum_{k=1}^{n} \Delta S_k 2\pi f(x_k)$$

From the last lecture, we know that

$$\Delta S_k = \Delta x_k \sqrt{1 + (\frac{\Delta y_k}{\Delta x_k})^2}$$

So:

$$S \cong \sum_{k=1}^{n} (\Delta x_k \sqrt{1 + (\frac{\Delta y_k}{\Delta x_k})^2}) 2\pi f(x_k)$$

Let $n \to \infty$ or $\Delta x_k \to 0$, then:

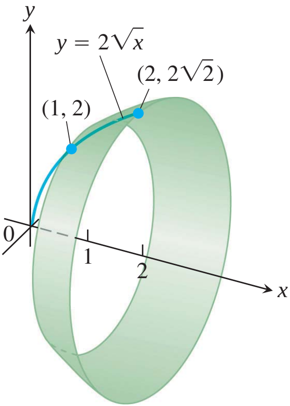
$$S = \lim_{n\to\infty} \sum_{k=1}^{n} (\Delta x_k \sqrt{1 + (\frac{\Delta y_k}{\Delta x_k})^2}) 2\pi f(x_k)$$

As this is a Riemann sum:

$$S = 2\pi \int_a^b f(x) \sqrt{1 + (\frac{dy}{dx})^2} dx$$

Example 2.44

Find the area of the surface generated by revolving the curve $y = 2\sqrt{x}$, $1 \leq x \leq 2$, about the $x-$axis



We evaluate the formula

$$S = \int_a^b 2\pi y\sqrt{1 + (\frac{dy}{dx})^2}$$

with:

$$a = 1, b = 2, y = 2\sqrt{x}, \quad \frac{dy}{dx} = \frac{1}{\sqrt{x}}$$

So:

$$\sqrt{1 + (\frac{dy}{dx})^2} = \sqrt{1 + (\frac{1}{\sqrt{x}})^2}$$

$$= \sqrt{1 + \frac{1}{x}} = \sqrt{\frac{x+1}{x}} = \frac{\sqrt{x+1}}{\sqrt{x}}$$

So:

$$S = \int_1^2 2\pi \cdot 2\sqrt{x} \cdot \frac{\sqrt{x+1}}{\sqrt{x}} dx = 4\pi \int_1^2 \sqrt{x+1} dx$$

$$= 4\pi \cdot \frac{2}{3}(x+1)^{3/2}\Big]_1^2 = \frac{8\pi}{3}(3\sqrt{3} - 2\sqrt{2})$$

Example 2.45

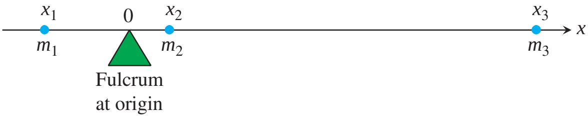
A reflector is formed by rotating $y = \sqrt{x}$ between $x = 0$ and $x = 1$ about the $x-$axis. What is the surface area.

Example 2.46

Find the area of the surface generated by revolving the curve $y = \frac{x^3}{9}$ between $x = 0$ and $x = 2$

Centers of Mass:

Centers of Mass:
Masses Along a Line:



Each mass m_k exerts a downward force $m_k g$.
Each of these forces has a tendency to turn the axis about the origin This turning effect is called a torque.
The torque of each mass is:

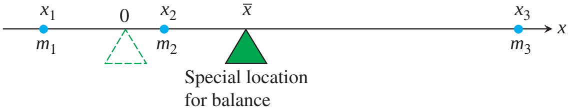
$$m_k g x_k$$

$$System \quad torque = m_1 g x_1 + m_2 g x_2 + m_3 g x_3$$

$$= g \cdot (m_1 x_1 + m_2 x_2 + m_3 x_3)$$

The number $m_1 x_1 + m_2 x_2 + m_3 x_3$ is called the moment of the system about the origin.

We usually want to know where to place the fulcrum to make the system balance, that is, at what point $\bar{x}$ to place it to make the torques add to zero.



The torque of each mass about the fulcrum in this special location is:

$$(x_k - \bar{x})m_k g$$

So the torque of the new system is

$$\sum_{k=1}^{3}(x_k - \bar{x})m_k g$$

Then:

$$\sum_{k=1}^{3}(x_k - \bar{x})m_k g = 0 \Rightarrow g\sum_{k=1}^{3}(x_k - \bar{x})m_k = 0$$

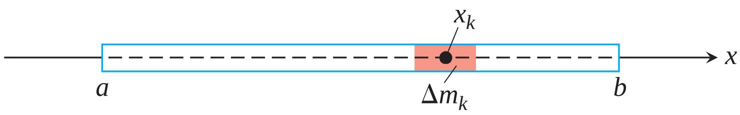
$$\Rightarrow \sum_{k=1}^{3}(m_k x_k - m_k\bar{x}) = 0 \Rightarrow \sum_{k=1}^{3} m_k x_k - \sum_{k=1}^{3} m_k\bar{x} = 0$$

$$\Rightarrow \sum_{k=1}^{3} m_k x_k = \bar{x}\sum_{k=1}^{3} m_k \Rightarrow \bar{x} = \frac{\sum_{k=1}^{3} m_k x_k}{\sum_{k=1}^{3} m_k}$$

This last equation tells us to find $\bar{x}$ by dividing the system's moment about the origin by the system's total mass:

$$\Rightarrow \bar{x} = \frac{\sum_{k=1}^{3} m_k x_k}{\sum_{k=1}^{3} m_k} = \frac{system\ \ moment\ \ about\ \ origin}{system\ \ mass}$$

Wires and Thin strips:



$$\bar{x} \cong \frac{system\ \ moment}{system\ \ mass}$$

The system mass is

$$\sum_{k=1}^{n} \Delta m_k$$

the moment of each piece $x_k \Delta m_k$, so

$$System\ \ moment \cong \sum_{k=1}^{n} x_k \Delta m_k$$

if the density of the strip at x_k is $\delta(x_k)$, expressed in terms of mass per unit length and if δ is continuous, then Δm_k is approximately equal to $\delta(x_k)\Delta x_k$:

$$\Delta m_k \cong \delta(x_k)\Delta x$$

Therefore:

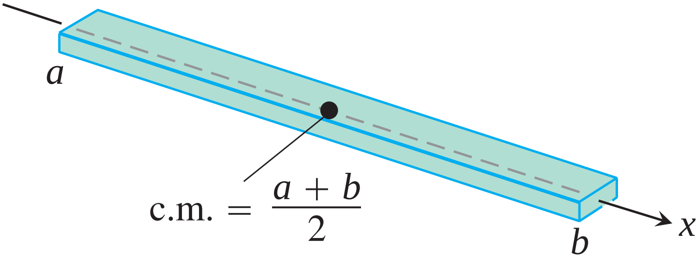
$$\bar{x} \cong \frac{system\ \ moment}{system\ \ mass} \cong \frac{\sum_{k=1}^{n} x_k \Delta m_k}{\sum_{k=1}^{n} \Delta m_k} \cong \frac{\sum_{k=1}^{n} x_k \delta(x_k) \Delta x}{\sum_{k=1}^{n} \delta(x_k) \Delta x}$$

The sums is a Riemann sum so when $\Delta x \to 0$ or $n \to \infty$, then

$$\bar{x} = \frac{\int_a^b x\delta(x)dx}{\int_a^b \delta(x)dx}$$

Example 2.47

Show that the center of mass of a straight, thin strip or rod of constant density lies halfway between its two ends.



We model the strip as a portion of the x-axis from $x = a$ to $x = b$. We know that

$$\bar{x} = \frac{\int_a^b x\delta(x)dx}{\int_a^b \delta(x)dx}$$

the density is constant so $\delta(x) = \delta$. The numerator is:

$$\int_a^b x\delta dx = \delta \int_a^b x dx = \delta \frac{1}{2}x^2 \Big]_a^b = \frac{\delta}{2}(b^2 - a^2)$$

The denominator is:

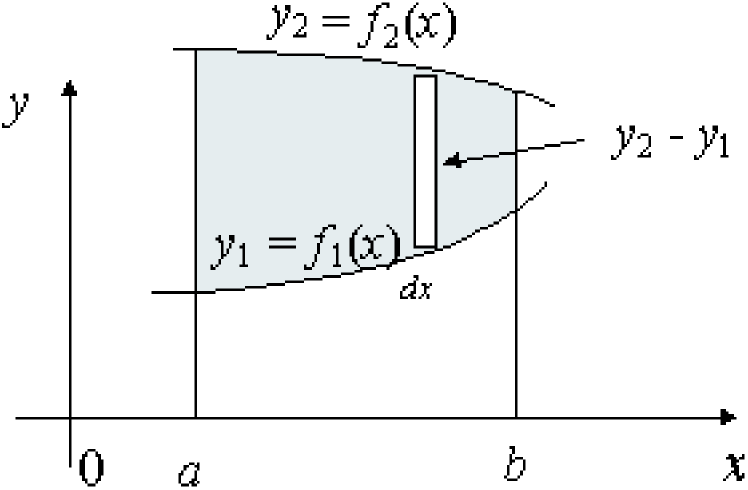
$$\int_a^b \delta dx = \delta \int_a^b dx = \delta x \Big]_a^b = \delta(b - a)$$

So:

$$\bar{x} = \frac{\int_a^b x\delta(x)dx}{\int_a^b \delta(x)dx} = \frac{a + b}{2}$$

When the density is constant, we call the center of mass the centroid of the object. To find the centroid we set $\delta = 1$

Centroid of a plane area:



Suppose the centroid of a typical strip is at (x_k, y_k) Then:

$$System\ \ mass \cong \sum_{k=1}^{n} \Delta m_k$$

The moments of the entire system about the two axes are:

$$Moment\ \ about\ \ x - axis \cong \sum_{k=1}^{n} y_k \Delta m_k$$

$$Moment\ \ about\ \ y - axis \cong \sum_{k=1}^{n} x_k \Delta m_k$$

The x-coordinate of the system's center of mass is defined to be:

$$\bar{x} = \frac{Moment\ \ about\ \ y - axis}{System\ \ mass} \cong \frac{\sum_{k=1}^{n} x_k \Delta m_k}{\sum_{k=1}^{n} \Delta m_k}$$

As $\delta = 1$, $\Delta m_k = \Delta A_k = (f_2(x_k) - f_1(x_k))\Delta x$. So:

$$\bar{x} \cong \frac{\sum_{k=1}^{n} x_k (f_2(x_k) - f_1(x_k))\Delta x}{\sum_{k=1}^{n} (f_2(x_k) - f_1(x_k))\Delta x}$$

The sums is a Riemann sum so when $\Delta x \to 0$ or $n \to \infty$, then

$$\bar{x} = \frac{\int_a^b x(f_2(x) - f_1(x))dx}{\int_a^b (f_2(x)) - f_1(x))dx} = \frac{\int_a^b x(f_2(x) - f_1(x))dx}{A}$$

The y-coordinate of the system's center of mass is defined to be:

$$\bar{y} = \frac{1}{2A} \int_a^b \left[(f_2(x))^2 - (f_1(x))^2 \right] dx$$

Example 2.48

Find the centroid of the area under the curve $y = \sqrt{x-2}$ between the lines $x = 2$ and $x = 5$

$$Area = \int_2^5 \sqrt{x-2}dx = \int_2^5 (x-2)^{1/2}dx = \frac{(x-2)^{3/2}}{3/2}\Big]_2^5$$

$$= \frac{2}{3}[(5-2)^{3/2} - (2-2)^{3/2}] = \frac{2}{3}3\sqrt{3} = 2\sqrt{3}$$

the x coordinate of the centroid is:

$$\bar{x} = \frac{1}{A}\int_2^5 x\sqrt{x-2}dx$$

Let $u = x-2$, so $x = u+2$ and $du = dx$ then:

$$\int x\sqrt{x-2}dx = \int (u+2)\sqrt{u}du = \int u^{3/2} + 2u^{1/2}du = \frac{u^{5/2}}{5/2} + 2\frac{u^{3/2}}{3/2} + c$$

$$= \frac{2}{5}(x-2)^{5/2} + \frac{4}{3}(x-2)^{3/2} + c$$

$$\bar{x} = \frac{1}{A}\int_2^5 x\sqrt{x-2}dx = \frac{1}{2\sqrt{3}}\Big(\frac{2}{5}(x-2)^{5/2} + \frac{4}{3}(x-2)^{3/2}\Big]_2^5\Big) = \frac{19}{5}$$

Also:

$$\bar{y} = \frac{1}{2A}\int_2^5 [f(x)]^2 dx = \frac{1}{2A}\int_2^5 (x-2)dx = \frac{1}{2A}(\frac{x^2}{2} - 2x\Big]_2^5)$$

So

$$\bar{y} = \frac{3\sqrt{3}}{8}$$

Average value of a function over a range:

The average of n numbers is:

$$\frac{1}{n}\sum_{i=1}^{n} x_i$$

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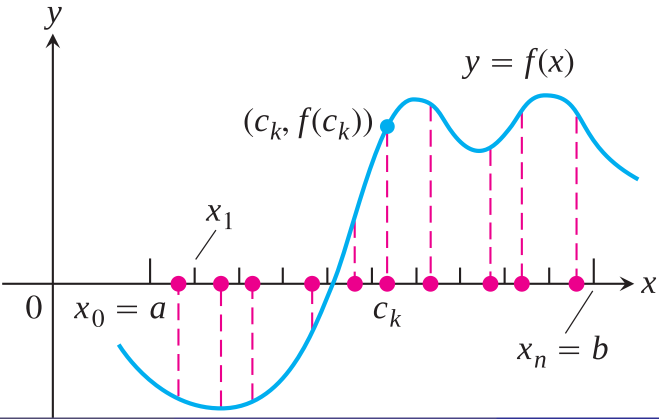


We divide $[a, b]$ into n subintervals of equal width $\Delta x = (b-a)/n$. The average of the n sampled values is:

$$\frac{f(c_1) + f(c_2) + \cdots + f(c_n)}{n} = \frac{1}{n}\sum_{k=1}^{n} f(c_k)$$

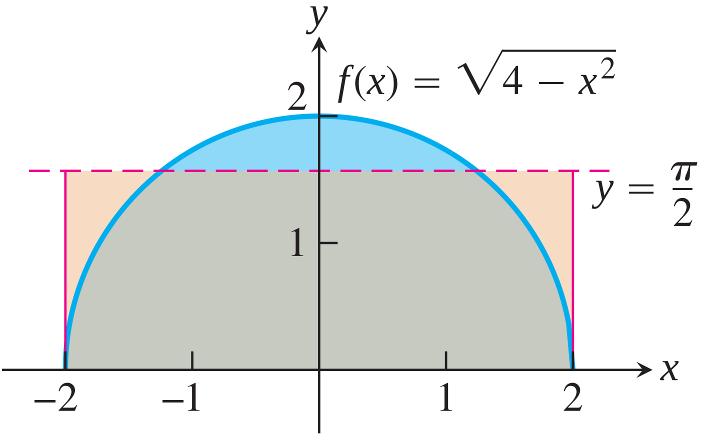
$$= \frac{\Delta x}{b-a}\sum_{k=1}^{n} f(c_k) = \frac{1}{b-a}\sum_{k=1}^{n} f(c_k)\Delta x$$

The average is obtained by dividing a Riemann sum for f on $[a, b]$ by $(b-a)$, so $n \to \infty$ or $\Delta x \to 0$, we get:

$$av(f) = \frac{1}{b-a}\int_a^b f(x)dx$$

Example 2.49

Find the average value of $f(x) = \sqrt{4 - x^2}$ on $[-2, 2]$



The average of the function is:

$$av(f) = \frac{1}{b-a}\int_a^b f(x)dx = \frac{Area}{b-a}$$

The area between the semicircle and the $x-$axis from -2 to 2 can be computed using the geometry formula:

$$Area = \frac{1}{2}\cdot \pi r^2 = \frac{1}{2}\cdot \pi(2)^2 = 2\pi$$

So the average value of f is

$$av(f) = \frac{1}{4}(2\pi) = \frac{\pi}{2}$$

It is sometimes convenient to use an alternative kind of average for the values of a function, $f(x)$, between $x = a$ and $x = b$.
The Root Mean Square Value provides a measure of central tendency for the numerical values of $f(x)$ and is defined to be the square root of the Mean Value of $f^2(x)$ from $x = a$ to $x = b$. Hence:

$$R.M.S = \sqrt{\frac{1}{b-a}\int_a^b [f(x)]^2 dx}$$

Example 2.50

An electric current $i(\theta)$ is given by $i(\theta) = I\sin(\theta)$ where I is a constant. Find R.M.S of $i(\theta)$ over $0 \leq \theta \leq 2\pi$

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$$R.M.S = \sqrt{\frac{1}{b-a}\int_a^b [f(x)]^2 dx}$$

So

$$R.M.S = \sqrt{\frac{1}{2\pi - 0}\int_0^{2\pi} [I\sin\theta]^2 d\theta}$$

Then:

$$[R.M.S]^2 = \frac{1}{2\pi - 0}\int_0^{2\pi} [I\sin\theta]^2 d\theta = \frac{I^2}{2\pi}\int_0^{2\pi} \sin^2\theta d\theta$$

Recall $\sin^2\theta = \frac{1}{2}(1-\cos 2\theta)$, so

$$[R.M.S]^2 = \frac{I^2}{2\pi}\int_0^{2\pi} \frac{1}{2}(1-\cos 2\theta)d\theta$$

So:

$$[R.M.S]^2 = \frac{I^2}{2\pi}\frac{1}{2}(\theta - \frac{1}{2}\sin 2\theta)\Big]_0^{2\pi}$$

Therefore

$$[R.M.S]^2 = \frac{I^2}{4\pi}\Big[(2\pi - \frac{1}{2}\sin(4\pi)) - (0 - \frac{1}{2}\sin(0))\Big]$$

Then:

$$[R.M.S]^2 = \frac{I^2}{4\pi}(2\pi) = \frac{I^2}{2} \Rightarrow R.M.S = \frac{I}{\sqrt{2}}$$

Example 2.51

Find the centroid of the area under the curve $y = 2x$ between the lines $x = 0$ and $x = 1$

Differential Equations:

Differential Equations:

Definition 2.52

An equation involving a derivative is a differential equation. For example

$$\frac{dy}{dt} = ky \quad (\star)$$

where k is some constant and y some function of t.

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So $y = e^{kt}$ is a solution of $(\star)$, also $y = 5e^{kt}$ satisfies the equation so we see that

$$y = Ae^{kt}$$

is a solution to $(\star)$ for any constant A

World Population:

Example 2.53

Suppose that the world population in 1960 was 3.06 billion, also suppose during this period the world population increased by 2% per year. Calculate today's population.

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In 1798 English economist Thomas Malthus suggested that the rate of change of a population is proportional to the size of the population.

Definition 2.54

Malthusian Law:

$$\frac{dy}{dt} = ky$$

This means that at time t the world population is

$$y = Ae^{kt}$$

for some constants A, k
Let's start time $t = 0$ in 1960, so

$$y(0) = 3.06 \ \ billion = Ae^{0k} = A \Rightarrow A = 3.06 \ \ billion$$

$$y(1) = 1.02y(0)$$
$$y(2) = 1.02y(1) = (1.02)^2 y(0)$$
$$y(3) = 1.02y(2) = (1.02)^3 y(0)$$

So in general:

$$y(t) = (1.02)^t y(0) \Rightarrow \frac{y(t)}{y(0)} = (1.02)^t = \frac{Ae^{kt}}{A} = e^{kt}$$

So $e^k = 1.02$, therefore:

$$\ln(e^k) = \ln(1.02) \Rightarrow k\ln(e) = \ln(1.02) \Rightarrow k = \ln(1.02) = 0.0198$$

So roughly

$$k = 0.02$$

Therefore:

$$y(t) = 3.06e^{0.02t}$$

Now let's calculate today's population using our model.

$$t = 2017 - 1960 = 57$$

So:

$$y = 3.06e^{0.02\times 57} = 9.38 \ \ billion$$

Definition 2.55

A **differential equation** is an equation containing derivatives (we often write **D.E.** for differential equation). For example:

- $\frac{dy}{dx} = 2xy$
- $\frac{d^2y}{dx^2} + 5\frac{dy}{dx} - 4y = e^x$

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Definition 2.56

The **order of a D.E.** is the order of the highest derivative in the D.E. For example:

- $\frac{dy}{dx} = 2xy$ is a first order differential equation
- $\frac{d^2y}{dx^2} + 5\frac{dy}{dx} - 4y = e^x$ is a second order differential equation

Separable Differential Equations:

Definition 2.57

A first order D.E. of the form

$$\frac{dy}{dx} = g(x)h(y)$$

is called a **separable D.E.**
(where $g(x)$ is a function of x and $h(y)$ is a function of y).

Separable Differential Equations:

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is called a **separable D.E.**
(where $g(x)$ is a function of x and $h(y)$ is a function of y).

This D.E. can be solved as follows:

$$\frac{1}{h(y)}\frac{dy}{dx} = g(x) \Rightarrow \int \frac{1}{h(y)}dy = \int g(x)dx$$

Then integrate and solve for y if possible.

Example 2.58

Solve the separable D.E.

$$\frac{dy}{dx} = \frac{3x^2}{\sin y}$$

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Solve the separable D.E.

$$\frac{dy}{dx} = \frac{3x^2}{\sin y}$$

$$\sin y dy = 3x^2 dx \Rightarrow \int \sin y dy = 3 \int x^2 dx$$

$$\Rightarrow -\cos y = x^3 + C \Rightarrow \cos y = -x^3 - C$$

$$\Rightarrow \cos^{-1}(\cos y) = \cos^{-1}(-x^3 - C) \Rightarrow y = \cos^{-1}(-x^3 - C)$$

Example 2.59

Solve the separable D.E.

$$\frac{dy}{dx} = 1 + y^2$$

Given that when $x = 0$ then $y = 0$

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Given that when $x = 0$ then $y = 0$

$$\frac{1}{1+y^2}\frac{dy}{dx} = 1 \Rightarrow \int \frac{1}{1+y^2}dy = \int 1dx$$

Let $y = \tan\theta$ so $dy = \sec^2\theta d\theta$. Therefore:

$$\int \frac{1}{1+\tan^2\theta}\sec^2\theta d\theta = \int dx$$

$$\Rightarrow \int d\theta = \int dx$$

So:

$$\theta = x + c$$

Now $x = 0, y = 0 \Rightarrow 0 = \tan\theta \Rightarrow \theta = 0$, thus

$$0 = 0 + c \Rightarrow c = 0$$

Therefore $\theta = x$ and $y = \tan x$ is our solution.

First Order Linear Differential Equations (The Integrating Factor Method):

Definition 2.60

A D.E. of the form

$$\frac{dy}{dx} + P(x)y = Q(x)$$

is called a **first order linear D.E.**, where $P(x)$ and $Q(x)$ are functions of x.

We can find the general solution of this D.E. as follows:

1.) Find the **integrating factor**

$$e^{\int P(x)dx}$$

2.) Multiply the D.E. by the integrating factor (I.F.) to get:

$$e^{\int P(x)dx}\left(\frac{dy}{dx} + P(x)y\right) = e^{\int P(x)dx}Q(x) \quad (\star)$$

3.) Note that the L.H.S. of ($\star$) equals:

$$\frac{d}{dx}(ye^{\int P(x)dx})$$

So ($\star$) becomes:

$$d(ye^{\int P(x)dx}) = e^{\int P(x)dx}Q(x)dx$$

4.) Integrate both sides and solve for y.
This algorithm is called the **integrating factor method**.

Example 2.61

Solve the first order linear differential equation

$$\frac{dy}{dx} - 3y = 0$$

using the integrating factor method.

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Solve the first order linear differential equation

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using the integrating factor method.

1.)

$$I.F. = e^{\int P(x)dx} = e^{\int -3dx} = e^{-3x}$$

(note that we do not include the arbitrary constant C).

2.) Multiply the D.E. by the I.F. to get

$$e^{-3x}(\frac{dy}{dx} - 3y) = 0e^{-3x} \quad (\star)$$

3.) Recall that the L.H.S. of $(\star)$ equals

$$\frac{d}{dx}(ye^{-3x})$$

therefore

$$\frac{d}{dx}(ye^{-3x}) = 0 \Rightarrow d(ye^{-3x}) = 0dx$$

4.) Integrate and solve for y:

$$\int d(ye^{-3x}) = \int 0dx \Rightarrow ye^{-3x} = 0 + C \Rightarrow y = Ce^{3x}$$

Last year exam Solutions:

Question $1.(a).i$:

$$\lim_{x\to 1}\frac{x-1}{(\sqrt{x}-1)(x+2)} = \lim_{x\to 1}\frac{(\sqrt{x}-1)(\sqrt{x}+1)}{(\sqrt{x}-1)(x+2)}$$
$$= \lim_{x\to 1}\frac{\sqrt{x}+1}{x+2} = \frac{2}{3}$$

Question $1.(a).ii$:

we apply L'Hôpital's Rule:

$$\lim_{\theta\to 0}\frac{6\sin\theta}{\theta+2\tan\theta} \overset{\mathrm{H}}{=} \lim_{\theta\to 0}\frac{6\cos\theta}{1+2\sec^2\theta} = \frac{6}{3} = 2$$

Question $2.(a).i$:

The derivative of the function $f(x)$ with respect to the variable x is the function f' or $\frac{df}{dx}$ whose value at x is

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

Question $2.(a).ii$:

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{(x+h)^2 + (x+h) - (x^2+x)}{h}$$

$$= \lim_{h\to 0} \frac{x^2 + h^2 + 2xh + x + h - x^2 - x}{h} = \lim_{h\to 0} \frac{h^2 + 2xh + h}{h}$$

$$= \lim_{h\to 0} (h + 2x + 1) = 2x + 1$$

Question $2.(b).i$:

$$f(x) = e^{\cos x^2} \sin x$$

$$\Rightarrow f'(x) = (e^{\cos x^2} \sin x)' = (e^{\cos x^2})' \sin x + e^{\cos x^2} (\sin x)' \quad (\star)$$

To differentiate $e^{\cos x^2}$ we use the chain rule:
Let $y = e^{\cos x^2} = e^u$, where $u = \cos x^2$, then by chain rule we have:

$$y' = \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = e^u \cdot (\cos x^2)'$$

To differentiate $\cos x^2$ we use the chain rule again:
$u = \cos x^2 = \cos v$, where $v = x^2$, then by chain rule we have:

$$u' = \frac{du}{dx} = \frac{du}{dv} \cdot \frac{dv}{dx} = (-\sin v)(2x) = (-\sin x^2)(2x)$$

So

$$(e^{\cos x^2})' = y' = e^u \cdot (-\sin x^2)(2x) = (e^{\cos x^2}) \cdot (-\sin x^2)(2x)$$

Therefore

$$({\color{red}\star}) = f'(x) = (e^{\cos x^2}) \cdot (-\sin x^2)(2x) \sin x + e^{\cos x^2}(\cos x)$$

Question $2.(b).iii$:

$$y = (\cos x)^x$$

$$\Rightarrow \ln y = \ln(\cos x)^x \Rightarrow \ln y = x \ln \cos x$$

$$\Rightarrow (\ln y)' = (x \ln \cos x)' \Rightarrow \frac{y'}{y} = 1 \cdot \ln \cos x + x \cdot \frac{(\cos x)'}{\cos x}$$

$$\Rightarrow \frac{y'}{y} = \ln \cos x - \frac{x \sin x}{\cos x}$$

$$\Rightarrow y' = y \Big[\ln \cos x - \frac{x \sin x}{\cos x} \Big] = (\cos x)^x \Big[\ln \cos x - \frac{x \sin x}{\cos x} \Big]$$